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A local existence theorem for the Einstein–Dirac equation

Eui Chul Kim

Department of Mathematics, Inha University, Inchon 402-751, South Korea

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Abstract

We studied the Einstein–Dirac equation as well as the weak Killing equation on Riemannian spin manifolds with codimension one foliation. We prove that, for any manifold M^n admitting real Killing spinors (resp. parallel spinors), there exist warped product metrics $\bar{\eta}$ on $M^n \times \mathbb{R}$ such that $(M^n \times \mathbb{R}, \bar{\eta})$ admit Einstein spinors (resp. weak Killing spinors). To prove the result we split the Einstein–Dirac equation into evolution equations and constraints, by means of Cartan's frame formalism, and apply the local preservation property of constraints.

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1. Introduction

Let (P^m, η) be an m-dimensional smooth oriented Riemannian spin manifold and denoted by Ric and S the Ricci tensor and the scalar curvature, respectively. Let $(,) = \mathrm{Re}\langle , \rangle$ be the real part of the standard Hermitian product $\langle , \rangle$ on the spinor bundle $\Sigma(P)$ over P^m. Let D be the Dirac operator acting on sections $\psi \in \Gamma(\Sigma(P))$ of the spinor bundle $\Sigma(P)$. The Einstein–Dirac equation is a minimal coupling of the Dirac equation to the Einstein equation and defined by (see [11])

$$D\psi = \lambda\psi, \qquad \mathrm{Ric} - \tfrac{1}{2} S\eta = T,$$

E-mail address: eckim@chollian.net (E.C. Kim).

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where $\lambda \in \mathbb{R}$ is some real number and the energy-momentum tensor T is given by

$$T(X, Y) = \pm\tfrac{1}{4}(X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi).$$

A non-trivial spinor field ψ solving this Einstein–Dirac system is called an *Einstein spinor* to eigenvalue $\lambda \in \mathbb{R}$. In case that the scalar curvature S does not vanish at any point, one defines the *weak Killing equation* by

$$\begin{aligned}\nabla_X \psi = {} & \frac{m}{2(m-1)S}\,\mathrm{d}S(X)\psi + \frac{1}{2(m-1)S} X \cdot \mathrm{d}S \cdot \psi \\ & + \frac{2\lambda}{(m-2)S}\mathrm{Ric}(X) \cdot \psi - \frac{\lambda}{m-2} X \cdot \psi,\end{aligned}$$

where $\lambda \in \mathbb{R}$ is some real number. A non-trivial solution ψ to the equation is called a *weak Killing spinor* to weak Killing number λ (shortly, WK-spinor to WK-number λ). Since rescaling the length of any WK-spinor provides an Einstein spinor, the WK-equation is stronger than the Einstein–Dirac equation (in dimension $n = 3$, the considered two equations are essentially equivalent). Moreover, the WK-equation reduces to the Killing equation [3,8]

$$\nabla_X \psi = -\frac{\lambda}{m} X \cdot \psi,$$

if the metric η is Einstein.

Till now, the known examples of the Einstein spinors on Riemannian manifolds are as follows:

(i) Real Killing spinors [2,3,9,15].
(ii) WK-spinors on quasi-Einstein Sasakian manifolds [11].
(iii) Einstein spinors on product manifolds $M^6 \times N^r$, where M^6 is a six-dimensional simply connected nearly Kähler manifold and N^r is a manifold of general dimension r admitting Killing spinors [11].
(iv) WK-spinors on the three-dimensional sphere S^3 with non-standard merics [4,10,11].
(v) WK-spinors on the three-dimensional Euclidean space $\mathbb{R}^3$ with non-constant scalar curvature [11,13].

The object of this paper is to establish a special existence theorem for WK as well as Einstein spinors. Namely, we prove the following theorem (see Theorems 5.1 and 7.1 and Corollary 7.1). Interestingly, we find that the WK-spinors constructed on $\mathbb{R}^3$ with non-constant scalar curvature (see [11, p. 171]) are a special case of this theorem.

Main Theorem. *Let (M^n, g_M) be a Riemannian manifold admitting a real Killing spinor ψ_M. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists a warped product metric $\bar{\eta}$ on $Q^{n+1} = M^n \times \mathbb{R}$ such that $(Q^{n+1}, \bar{\eta})$ admits an Einstein spinor ψ to eigenvalue λ_Q. In particular, if ψ_M is a parallel spinor, then the Einstein spinor ψ becomes a WK-spinor to WK-number λ_Q.*

The key idea to prove the theorem is to split the Einstein–Dirac equation into evolution equations and constraints and apply the local preservation property of the constraints.

We will explicitly give an initial-value formulation for the Einstein–Dirac equation, in Riemannian setting, and solve it for a specific class of initial data sets. It is well-known that, in Riemannian signature the Einstein equations are generally of elliptic type, making the initial-value problem (the Cauchy problem) for general smooth data inappropriate. However, when the considered Riemannian manifolds admit a codimension one foliation, one can represent the Einstein equations to be of hyperbolic type, just as one does over Lorentzian manifolds, and can indeed formulate the initial-value problem in a natural way.

So far, not much has been studied about the initial-value problem for the Einstein–Dirac equation. In Lorentzian signature, the spacelike initial-value problem for the Einstein–Dirac system was considered by Bao et al. [1] in terms of $3+1$ Hamiltonian formalism, but no existence theorem was proved there. Recently, Friedrich and Rendall [7] indicated, in terms of Penrose's two-spinor formalism, that the Einstein–Dirac equation may be reduced to symmetric hyperbolic evolution equations, illustrating some questions arising in the reduction.

In this paper we give an invariant description of the initial-value formulation for the Einstein–Dirac equation on Riemannian manifolds with codimension one foliation, in an explicit form and in complete generality. The splitting of the Einstein–Dirac equation into evolution equations and constraints will be achieved in terms of Cartan's frame formalism, and hence our formulation is valid on Riemannian manifolds $M^n \times \mathbb{R}$ of general dimension $n+1$. Sections 2–4 of this paper are devoted to establishing the basic framework, the *hyperbolic representation* of curvatures and the Dirac equation, on (possibly compact) manifolds with codimension one foliation, and the framework may be of independent interest for further study of the behaviour of spinor field equations under global change of metrics.

2. Representation of curvatures and the Dirac equation with respect to reference metric

Let P^m be an m-dimensional simply connected smooth oriented manifold allowing spin structure, and let $\eta, \bar{\eta}$ be two Riemannian metrics on P^m. Henceforth we fix the notation η to denote a reference metric. Then there exists a unique $(1,1)$-tensor field K on P^m that is positive definite with respect to η and satisfies

$$\bar{\eta}(X, Y) = \eta(K(X), K(Y))$$

for all vector fields X, Y. Recall that the Levi-Civita connection $\nabla^{\bar{\eta}}$ of $(P^m, \bar{\eta})$ is characterized by the Koszul formula

$$2\bar{\eta}(Z, \nabla^{\bar{\eta}}_X Y) = X\{\bar{\eta}(Y, Z)\} + Y\{\bar{\eta}(Z, X)\} - Z\{\bar{\eta}(X, Y)\} + \bar{\eta}(Z, [X, Y]) \\ + \bar{\eta}(Y, [Z, X]) - \bar{\eta}(X, [Y, Z]).$$

Letting $(E_1, \ldots, E_m)$ be a local η-orthonormal frame field on P^m, for which

$$(F_1 = K^{-1}(E_1), \ldots, F_m = K^{-1}(E_m)),$$

is $\bar{\eta}$-orthonormal, and inserting $X = F_i, Y = F_j, Z = F_k$ into the Koszul formula, we have

$$\begin{aligned}
&2\eta(E_k, K\{\nabla^{\bar{\eta}}_{K^{-1}(E_i)}(K^{-1}E_j)\}) \\
&\quad = \eta(E_k, K\{[K^{-1}(E_i), K^{-1}(E_j)]\}) + \eta(E_j, K\{[K^{-1}(E_k), K^{-1}(E_i)]\}) \\
&\qquad - \eta(E_i, K\{[K^{-1}(E_j), K^{-1}(E_k)]\}) \\
&\quad = \eta(E_k, K(\nabla^{\eta}_{F_i}F_j) - K(\nabla^{\eta}_{F_j}F_i)) + \eta(E_j, K(\nabla^{\eta}_{F_k}F_i) - K(\nabla^{\eta}_{F_i}F_k)) \\
&\qquad - \eta(E_i, K(\nabla^{\eta}_{F_j}F_k) - K(\nabla^{\eta}_{F_k}F_j)) \\
&\quad = \eta(E_k, K\{(\nabla^{\eta}_{F_i}K^{-1})(E_j)\} + \nabla^{\eta}_{F_i}E_j - K\{(\nabla^{\eta}_{F_j}K^{-1})(E_i)\} - \nabla^{\eta}_{F_j}E_i) \\
&\qquad + \eta(E_j, K\{(\nabla^{\eta}_{F_k}K^{-1})(E_i)\} + \nabla^{\eta}_{F_k}E_i - K\{(\nabla^{\eta}_{F_i}K^{-1})(E_k)\} - \nabla^{\eta}_{F_i}E_k) \\
&\qquad - \eta(E_i, K\{(\nabla^{\eta}_{F_j}K^{-1})(E_k)\} + \nabla^{\eta}_{F_j}E_k - K\{(\nabla^{\eta}_{F_k}K^{-1})(E_j)\} - \nabla^{\eta}_{F_k}E_j) \\
&\quad = 2\eta(E_k, \nabla^{\eta}_{K^{-1}(E_i)}E_j) + \eta(E_k, K\{(\nabla^{\eta}_{K^{-1}(E_i)}K^{-1})(E_j)\} \\
&\qquad - K\{(\nabla^{\eta}_{K^{-1}(E_j)}K^{-1})(E_i)\}) + \eta(E_j, K\{(\nabla^{\eta}_{K^{-1}(E_k)}K^{-1})(E_i)\} \\
&\qquad - K\{(\nabla^{\eta}_{K^{-1}(E_i)}K^{-1})(E_k)\}) - \eta(E_i, K\{(\nabla^{\eta}_{K^{-1}(E_j)}K^{-1})(E_k)\} \\
&\qquad - K\{(\nabla^{\eta}_{K^{-1}(E_k)}K^{-1})(E_j)\}).
\end{aligned}$$

Thus we obtain the following formula.

Proposition 2.1. *The Levi-Civita connections* $\nabla^{\bar{\eta}}, \nabla^{\eta}$ *are related by*

$$\nabla^{\bar{\eta}}_{K^{-1}(X)}(K^{-1}(Y)) = K^{-1}(\nabla^{\eta}_{K^{-1}(X)}Y) + K^{-1}\{\Lambda_\eta(X, Y)\},$$

where Λ_η *is the* $(1, 2)$*-tensor field defined by*

$$\begin{aligned}
2\eta(\Lambda_\eta(X, Y), Z) = {}& \eta(Z, K\{(\nabla^{\eta}_{K^{-1}(X)}K^{-1})(Y)\} - K\{(\nabla^{\eta}_{K^{-1}(Y)}K^{-1})(X)\}) \\
&+ \eta(Y, K\{(\nabla^{\eta}_{K^{-1}(Z)}K^{-1})(X)\} - K\{(\nabla^{\eta}_{K^{-1}(X)}K^{-1})(Z)\}) \\
&+ \eta(X, K\{(\nabla^{\eta}_{K^{-1}(Z)}K^{-1})(Y)\} - K\{(\nabla^{\eta}_{K^{-1}(Y)}K^{-1})(Z)\}).
\end{aligned}$$

Remark 2.1.

(i) The exact difference between the Levi-Civita connections $\nabla^{\bar{\eta}}, \nabla^{\eta}$ is related to the tensor Λ_η by

$$\nabla^{\bar{\eta}}_X Y - \nabla^{\eta}_X Y = K^{-1}\{\Lambda_\eta(KX, KY)\} + K^{-1}\{(\nabla^{\eta}_X K)(Y)\}.$$

(ii) The relation

$$\eta(\Lambda_\eta(X, Z), Y) + \eta(\Lambda_\eta(X, Y), Z) = 0,$$

is valid for all vector fields X, Y, Z.

(iii) Since

$$\Lambda_\eta(X,Y)-\Lambda_\eta(Y,X)=K\{(\nabla^\eta_{K^{-1}X}K^{-1})(Y)\}-K\{(\nabla^\eta_{K^{-1}Y}K^{-1})(X)\}$$

$\Lambda_\eta \equiv 0$ vanishes identically if and only if $\Lambda_\eta(X,Y)=\Lambda_\eta(Y,X)$ for all vector fields X, Y.

We will often use the shorthand notation $\Lambda_\eta=\Lambda$, if there is no possibility of confusion. Proposition 2.1 enables us to describe the behaviour of curvatures under global change of metrics in a nice way: a direct computation gives

$$\begin{aligned}&R_{\bar\eta}(K^{-1}X,K^{-1}Z)(K^{-1}Y)-K^{-1}\{R_\eta(K^{-1}X,K^{-1}Z)(Y)\}\\&\quad=K^{-1}\{(\nabla^\eta_{K^{-1}(X)}\Lambda)(Z,Y)-(\nabla^\eta_{K^{-1}(Z)}\Lambda)(X,Y)\}+K^{-1}\{\Lambda(X,\Lambda(Z,Y))\\&\qquad-\Lambda(Z,\Lambda(X,Y))\}+K^{-1}\{\Lambda(\Lambda(Z,X)-\Lambda(X,Z),Y)\},\end{aligned}$$

where $R_{\bar\eta}$ (resp. R_η) is the Riemann tensor of $\bar\eta$ (resp. η). Contracting both sides of the equation, we can now represent the Ricci curvature $\mathrm{Ric}_{\bar\eta}$ as well as the scalar curvature $S_{\bar\eta}$ with respect to the reference metric η.

Proposition 2.2.

$$\begin{aligned}&\mathrm{Ric}_{\bar\eta}(K^{-1}Y,K^{-1}Z)-\sum_{j=1}^m\eta(E_j,R_\eta(K^{-1}E_j,K^{-1}Z)(Y))\\&\quad=\sum_{j=1}^m\eta(E_j,(\nabla^\eta_{K^{-1}E_j}\Lambda)(Z,Y)-(\nabla^\eta_{K^{-1}Z}\Lambda)(E_j,Y))+\sum_{j=1}^m\eta(E_j,\Lambda(E_j,\Lambda(Z,Y))\\&\qquad-\Lambda(\Lambda(E_j,Z),Y)).\end{aligned}$$

In particular,

$$\begin{aligned}&S_{\bar\eta}-\sum_{i,j=1}^m\eta(E_i,R_\eta(K^{-1}E_i,K^{-1}E_j)(E_j))\\&\quad=2\sum_{i,j=1}^m\eta(E_i,(\nabla^\eta_{K^{-1}(E_i)}\Lambda)(E_j,E_j))-\sum_{i,j,k=1}^m\eta(E_k,\Lambda(E_i,E_i))\eta(E_k,\Lambda(E_j,E_j))\\&\qquad-\sum_{i,j,k=1}^m\eta(E_k,\Lambda(E_i,E_j))\eta(E_k,\Lambda(E_j,E_i)).\end{aligned}$$

Next, we review briefly the behaviour of the Dirac operator under change of metrics. Let $T(P)$ be the tangent bundle of P^m, and let $\Sigma(P)_{\bar\eta}$ (resp. $\Sigma(P)_\eta$) be the spinor bundle of $(P,\bar\eta)$ (resp. (P,η)) equipped with the standard Hermitian product $\langle,\rangle_{\bar\eta}$ (resp. $\langle,\rangle_\eta$). We know that there exists a natural isomorphism $\tilde K:\Sigma(P)_{\bar\eta}\to\Sigma(P)_\eta$ with

$$\langle\tilde K(\varphi),\tilde K(\psi)\rangle_\eta=\langle\varphi,\psi\rangle_{\bar\eta},\qquad (KZ)\cdot(\tilde K\psi)=\tilde K(Z\cdot\psi),$$

for all $Z \in T(P), \varphi, \psi \in \Sigma(P)_{\bar{\eta}}$, where the dots "·" in the latter relation indicate the Clifford multiplication with respect to η and $\bar{\eta}$, respectively. In terms of local $\bar{\eta}$-orthonormal frame field $(F_1, \dots, F_m)$, the spin derivative $\nabla^{\bar{\eta}}\varphi$ is expressed as

$$\nabla^{\bar{\eta}}_X\varphi = X(\varphi) + \frac{1}{4}\sum_{i=1}^{m} F_i \cdot \nabla^{\bar{\eta}}_X F_i \cdot \varphi, \quad \varphi \in \Gamma(\Sigma(P)_{\bar{\eta}}),$$

and the Dirac operator $D_{\bar{\eta}}\varphi$ as

$$D_{\bar{\eta}}\varphi = \sum_{j=1}^{m} F_j \cdot \nabla^{\bar{\eta}}_{F_j}\varphi.$$

Making use of the formula in Proposition 2.1, one finds now readily that the spinor derivatives $\nabla^{\bar{\eta}}, \nabla^{\eta}$ and the Dirac operators $D_{\bar{\eta}}, D_{\eta}$ are related as follows.

Proposition 2.3. (see [5]). *For all* $\psi \in \Gamma(\Sigma(P)_{\eta})$

$$\{\tilde{K} \circ \nabla^{\bar{\eta}}_{K^{-1}(E_j)} \circ (\tilde{K})^{-1}\}(\psi) = \nabla^{\eta}_{K^{-1}(E_j)}\psi + \frac{1}{4}\sum_{k,l=1}^{m} \Lambda_{jkl}E_k \cdot E_l \cdot \psi,$$

$$\begin{aligned}\{\tilde{K} \circ D_{\bar{\eta}} \circ (\tilde{K})^{-1}\}(\psi) &= \sum_{i=1}^{m} E_i \cdot \nabla^{\eta}_{K^{-1}(E_i)}\psi + \frac{1}{4}\sum_{j,k,l=1}^{m} \Lambda_{jkl}E_j \cdot E_k \cdot E_l \cdot \psi \\ &= \sum_{i=1}^{m} E_i \cdot \nabla^{\eta}_{K^{-1}(E_i)}\psi - \frac{1}{2}\sum_{j,k=1}^{m} \Lambda_{jjk}E_k \cdot \psi \\ &\quad + \frac{1}{2}\sum_{j<k<l}^{m} (\Lambda_{jkl} + \Lambda_{klj} + \Lambda_{ljk})E_j \cdot E_k \cdot E_l \cdot \psi,\end{aligned}$$

where $\Lambda_{jkl} := \eta(\Lambda_{\eta}(E_j, E_k), E_l)$.

3. Representation of curvatures on manifolds with codimension one foliation

In this section we establish an intrinsic setting of the formulas that constitute the well-known evolution system for the Einstein (vacuum) equation (see [6,7]). The evolution system consists of two differential equations, describing the evolution of metrics (see Corollary 3.1) and the evolution of symmetric $(0, 2)$-tensor fields (see Proposition 3.2), respectively. The main aim of this section is to represent the Ricci tensor $\mathrm{Ric}_{\bar{\eta}}$ *hyperbolically* with respect to codimension one foliation. We use the terminology "*hyperbolic* representation" in the sense that such representation of differential operators, on manifolds with codimension one foliation, transforms field equations of elliptic type involving metrics to hyperbolic systems in PDE theory. Note in this view that the formulas in Proposition 2.2 may be thought of as the *elliptic* representation of curvatures.

Let (Q^{n+1}, η) be an $(n+1)$-dimensional smooth oriented Riemannian spin manifold. We assume that there exists a codimension one foliation on (Q^{n+1}, η) defined by a unit vector field E_{n+1} with $\mathrm{d}E^{n+1} = 0$, where $E^{n+1} = \eta(E_{n+1}, \cdot)$ is the dual 1-form of E_{n+1}. Letting $E_{n+1}^{\perp}$ denote the η-orthogonal complement of E_{n+1} in the tangent bundle $T(Q)$, we note that $\mathrm{d}E^{n+1} = 0$ implies the following facts (e.g. see [14]):

(i) For all vector fields V, W belonging to $E_{n+1}^{\perp}$, all of $[V, W]$, $\nabla^{\eta}_{E_{n+1}} V$ and $\nabla^{\eta}_{W} E_{n+1}$ belong to $E_{n+1}^{\perp}$.
(ii) $\nabla^{\eta}_{E_{n+1}} E_{n+1} = 0$.
(iii) If Q^{n+1} is compact, then all the slices of the foliation are diffeomorphic.
(iv) If Q^{n+1} is simply connected, then $E^{n+1} = \mathrm{d}s$ for some real-valued function $s : Q^{n+1} \to \mathbb{R}$ (Q^{n+1} must be non-compact) and the foliation is defined by the level hypersurfaces $s = \text{constant}$.

Let $(E_1, \dots, E_n, E_{n+1})$ be a local η-orthonormal frame field on Q^{n+1}, with $E_j \in E_{n+1}^{\perp}$, $j = 1, \dots, n$, and $(E^1, \dots, E^n, E^{n+1})$ the dual frame field. Denote $\otimes^r_s(E_{n+1}^{\perp})$ the space of all (r, s)-tensor fields B on Q^{n+1} such that

$$\eta(E_{i_1} \otimes \cdots \otimes E_{i_r}, B(E_{j_1} \otimes \cdots \otimes E_{j_s})) = 0,$$

whenever either $i_k = n+1$ for some i_k or $j_l = n+1$ for some j_l. Now, consider a positive definite $(1, 1)$-tensor field K on (Q^{n+1}, η). Letting $\bar{\eta}$ be the metric induced by K via $\bar{\eta}(X, Y) = \eta(K(X), K(Y))$ and identifying $\bar{\eta}$ with K^2, we can express $\bar{\eta}$ as

$$\bar{\eta} = K^2 = \left\{ \sum_{i,j=1}^{n} (L^2)^i_j E^j \otimes E_i \right\} + E^{n+1} \otimes L^2(\zeta) + \eta(L^2(\zeta), \cdot) \otimes E_{n+1}$$
$$+ \{\eta(L(\zeta), L(\zeta)) + \rho^2\} E^{n+1} \otimes E_{n+1},$$

where $L \in \otimes^1_1(E_{n+1}^{\perp})$, $\zeta \in \otimes^1_0(E_{n+1}^{\perp})$ and $\rho : Q^{n+1} \to \mathbb{R}$ is a positive function.

This may be thought of as an intrinsic (Riemannian) version of the well-known ADM-representation of metrics in general relativity. ζ agrees with the *shift vector field* and ρ with the *lapse function*. Note that the $(1, 1)$-tensor K^2 is related to the $(1, 1)$-tensor L^2 by

$$K^2(V) = L^2(V) + \eta(L^2(\zeta), V) E_{n+1}$$

for all vector fields $V \in \otimes^1_0(E_{n+1}^{\perp})$ and

$$K^2(E_{n+1}) = L^2(\zeta) + \{\eta(L(\zeta), L(\zeta)) + \rho^2\} E_{n+1}.$$

Furthermore, L^2 is positive definite on each slice of the foliation and, on the slices, coincides with the metrics induced by K^2. Certainly

$$F_1 := L^{-1}(E_1), \dots, F_n := L^{-1}(E_n),\ F_{n+1} := \rho^{-1}(E_{n+1} - \zeta),$$

is a local $\bar{\eta}$-orthonormal frame field on Q^{n+1}, its dual frame field being given by

$$F^i = L(E^i) + \eta(L(\zeta), E_i) E^{n+1}, \qquad F^{n+1} = \rho E^{n+1}.$$

Let Z be a vector field on Q^{n+1}, and let V, W be vector fields in $E_{n+1}^{\perp}$. In what follows we fix the notations V, W to mean vector fields in $E_{n+1}^{\perp}$. Then, one verifies easily the following basic identities:

$$\begin{aligned}
&\bar{\eta}(V, W) = \eta(L^2(V), W) = \eta(V, L^2(W)) = \eta(L(V), L(W)),\\
&\bar{\eta}(V, E_{n+1}) = \eta(V, L^2(\zeta)), \qquad \bar{\eta}(E_{n+1}, E_{n+1}) = \eta(L(\zeta), L(\zeta)) + \rho^2,\\
&\bar{\eta}(Z, F_{n+1}) = \rho\eta(Z, E_{n+1}), \qquad \bar{\eta}(V, F_{n+1}) = 0.
\end{aligned}$$

The identity $\bar{\eta}(V, F_{n+1}) = 0$ in the last line implies that $E_{n+1}^{\perp}$ coincides with the $\bar{\eta}$-orthogonal complement of F_{n+1} in $T(Q)$.

Let

$$\mathrm{II}(V) := -\nabla_V^{\bar{\eta}} F_{n+1} \quad \text{and} \quad \Theta(V) := -\nabla_V^{\eta} E_{n+1},$$

denote the second fundamental form, on each slice, defined by the unit vector field F_{n+1} and E_{n+1}, respectively. Let $\bar{g}$ (resp. g) denote the metric, on each slice, induced by $\bar{\eta}$ (resp. η) and $\nabla^{\bar{g}}$ (resp. ∇^{g}) its Levi-Civita connection. In the notations, the tensor L satisfies

$$\begin{aligned}
&\nabla_{E_{n+1}}^{\eta} L \in \otimes_1^1(E_{n+1}^{\perp}), \qquad (\nabla_V^{\eta} L)(W) = (\nabla_V^{g} L)(W) + \Theta(V, L(W))E_{n+1},\\
&(\nabla_V^{\eta} L)(E_{n+1}) = L(\Theta(V)).
\end{aligned}$$

In order to represent curvatures of $\bar{\eta}$ hyperbolically with respect to codimension one foliation (Proposition 3.2), we must explicitly know how the connection $\nabla^{\bar{\eta}}$ is related to the connections ∇^{η} and ∇^{g}. This is done in the following proposition, which may be thought of as the hyperbolic version of Proposition 2.1.

Proposition 3.1.

$$\begin{aligned}
\text{(i)}\quad \nabla_{F_i}^{\bar{\eta}} F_j &= \nabla_{F_i}^{\bar{g}} F_j + \mathrm{II}(F_i, F_j)F_{n+1} \quad (1 \le i, j \le n) = L^{-1}(\nabla_{L^{-1}E_i}^{g} E_j)\\
&\quad + L^{-1}\{\Lambda_g(E_i, E_j)\} + \mathrm{II}(L^{-1}E_i, L^{-1}E_j)\{\rho^{-1}(E_{n+1} - \zeta)\},
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad \rho \cdot \nabla_{F_{n+1}}^{\bar{\eta}} F_j &= L^{-1}(\nabla_{E_{n+1}}^{\eta} E_j) - L^{-1}(\nabla_{\zeta}^{g} E_j) + \frac{1}{2}(\nabla_{E_{n+1}}^{\eta} L^{-1})(E_j)\\
&\quad - \frac{1}{2}(\nabla_{\zeta}^{g} L^{-1})(E_j) + \frac{1}{2}\nabla_{L^{-1}E_j}^{g}\zeta\\
&\quad + \frac{1}{2}\Theta(L^{-1}E_j) + \frac{1}{2}\sum_{i=1}^{n} \eta(E_j, (\nabla_{E_{n+1}}^{\eta} L)(L^{-1}E_i))L^{-1}E_i\\
&\quad - \frac{1}{2}\sum_{i=1}^{n} \eta(E_j, (\nabla_{\zeta}^{g} L)(L^{-1}E_i) + L(\nabla_{L^{-1}E_i}^{g}\zeta)\\
&\quad + (L \circ \Theta \circ L^{-1})(E_i))L^{-1}E_i + \mathrm{d}\rho(L^{-1}E_j)F_{n+1},
\end{aligned}$$

(iii) $$-\rho\cdot\nabla^{\bar\eta}_{F_j}F_{n+1}=\rho\cdot \mathrm{II}(L^{-1}E_j)$$
$$=\frac{1}{2}(\nabla^{\eta}_{E_{n+1}}L^{-1})(E_j)-\frac{1}{2}\sum_{i=1}^{n}\eta(E_j,(\nabla^{\eta}_{E_{n+1}}L)(L^{-1}E_i))L^{-1}E_i$$
$$-\frac{1}{2}(\nabla^{g}_{\zeta}L^{-1})(E_j)+\frac{1}{2}\nabla^{g}_{L^{-1}E_j}\zeta+\frac{1}{2}\Theta(L^{-1}E_j)$$
$$+\frac{1}{2}\sum_{i=1}^{n}\eta(E_j,(\nabla^{g}_{\zeta}L)(L^{-1}E_i)+L(\nabla^{g}_{L^{-1}E_i}\zeta)$$
$$+(L\circ\Theta\circ L^{-1})(E_i))L^{-1}E_i,$$

(iv) $$\nabla^{\bar\eta}_{F_{n+1}}F_{n+1}=-\rho^{-1}\sum_{i=1}^{n}\mathrm{d}\rho(L^{-1}E_i)L^{-1}E_i.$$

Proof. One computes directly, substituting the identities

$$[F_{n+1},V]=\nabla^{\bar\eta}_{F_{n+1}}V+\mathrm{II}(V)=\rho^{-1}\mathrm{d}\rho(V)F_{n+1}+\rho^{-1}[E_{n+1},V]+\rho^{-1}[V,\zeta]$$
$$=\rho^{-1}\mathrm{d}\rho(V)F_{n+1}+\rho^{-1}\nabla^{\eta}_{E_{n+1}}V+\rho^{-1}\Theta(V)+\rho^{-1}[V,\zeta],$$

in the Koszul formula. □

We can equivalently rewrite the third equation (iii) in Proposition 3.1 as follows.

Corollary 3.1.

$$2\rho\cdot\mathrm{II}(V,W)=-\eta((\nabla^{\eta}_{E_{n+1}}L)(V),L(W))-\eta(L(V),(\nabla^{\eta}_{E_{n+1}}L)(W))$$
$$+\eta((\nabla^{g}_{\zeta}L)(V),L(W))+\eta(L(V),(\nabla^{g}_{\zeta}L)(W))$$
$$+\eta(\nabla^{g}_{V}\zeta+\Theta(V),L^2(W))+\eta(L^2(V),\nabla^{g}_{W}\zeta+\Theta(W)).$$

To prove Proposition 3.2 below, we need to recall the Gauss–Codazzi equations that relates the curvatures of $(Q^{n+1},\bar\eta)$ to the curvatures of the slices.

Lemma 3.1.

(i) $$R_{\bar\eta}(V_1,V_2)(V_3)-R_{\bar g}(V_1,V_2)(V_3)$$
$$=\mathrm{II}(V_1,V_3)\mathrm{II}(V_2)-\mathrm{II}(V_2,V_3)\mathrm{II}(V_1)+\{(\nabla^{\bar g}_{V_1}\mathrm{II})(V_2,V_3)$$
$$-(\nabla^{\bar g}_{V_2}\mathrm{II})(V_1,V_3)\}F_{n+1},$$

(ii) $$\mathrm{Ric}_{\bar\eta}(W)-R_{\bar\eta}(W,F_{n+1})(F_{n+1})-\mathrm{Ric}_{\bar g}(W)$$
$$=(\mathrm{II}\circ\mathrm{II})(W)-(\mathrm{Tr}_{\bar g}\mathrm{II})\mathrm{II}(W)+\{\mathrm{d}(\mathrm{Tr}_{\bar g}\mathrm{II})(W)-(\mathrm{div}_{\bar g}\mathrm{II})(W)\}F_{n+1},$$

(iii) $$S_{\bar\eta}-2\cdot\mathrm{Ric}_{\bar\eta}(F_{n+1},F_{n+1})-S_{\bar g}=\mathrm{Tr}_{\bar g}(\mathrm{II}^2)-(\mathrm{Tr}_{\bar g}\mathrm{II})^2.$$

Proposition 3.2.

$$(\nabla^{\eta}_{E_{n+1}}\mathrm{II})(V,W)=\rho\cdot\{\mathrm{Ric}_{\bar\eta}(V,W)-\mathrm{Ric}_{\bar g}(V,W)-2\bar\eta(V,\mathrm{II}^2(W))+(\mathrm{Tr}_{\bar g}\mathrm{II})\mathrm{II}(V,W)\}\\+(\nabla^{\bar g}_{\zeta}\mathrm{II})(V,W)+\bar\eta(\mathrm{II}(V),\nabla^{\bar g}_{W}\zeta)+\bar\eta(\mathrm{II}(W),\nabla^{\bar g}_{V}\zeta)\\+\bar\eta(\Theta(V),\mathrm{II}(W))+\bar\eta(\Theta(W),\mathrm{II}(V))+(\nabla^{\bar g}_{W}\,\mathrm{d}\rho)(V).$$

Proof. Via a direct computation, we have

$$R_{\bar\eta}(W,F_{n+1})(F_{n+1})=\nabla^{\bar g}_{W}(\nabla^{\bar\eta}_{F_{n+1}}F_{n+1})+\rho^{-1}\,\mathrm{d}\rho(W)\nabla^{\bar\eta}_{F_{n+1}}F_{n+1}-(\mathrm{II}\circ\mathrm{II})(W)\\+\rho^{-1}[E_{n+1},\mathrm{II}(W)]-\rho^{-1}[\zeta,\mathrm{II}(W)]-\rho^{-1}\mathrm{II}([E_{n+1},W])\\-\rho^{-1}\mathrm{II}([W,\zeta]).$$

Using the equation (iv) in Proposition 3.1, we compute

$$\bar\eta(V,R_{\bar\eta}(W,F_{n+1})(F_{n+1}))\\=W\{\bar\eta(V,\nabla^{\bar\eta}_{F_{n+1}}F_{n+1})\}-\bar\eta(\nabla^{\bar g}_{W}V,\nabla^{\bar\eta}_{F_{n+1}}F_{n+1})+\rho^{-1}\,\mathrm{d}\rho(W)\bar\eta(V,\nabla^{\bar\eta}_{F_{n+1}}F_{n+1})\\-\bar\eta(V,\mathrm{II}^2(W))+\rho^{-1}\bar\eta(V,[E_{n+1},\mathrm{II}(W)])-\rho^{-1}\bar\eta(V,[\zeta,\mathrm{II}(W)])\\-\rho^{-1}\bar\eta(\mathrm{II}(V),[E_{n+1},W])+\rho^{-1}\bar\eta(\mathrm{II}(V),[\zeta,W])\\=\rho^{-2}\,\mathrm{d}\rho(V)\,\mathrm{d}\rho(W)-\rho^{-1}(\nabla^{\bar g}_{W}\,\mathrm{d}\rho)(V)-\rho^{-1}\,\mathrm{d}\rho(\nabla^{\bar g}_{W}V)+\rho^{-1}\,\mathrm{d}\rho(\nabla^{\bar g}_{W}V)\\-\rho^{-2}\,\mathrm{d}\rho(V)\,\mathrm{d}\rho(W)-\bar\eta(V,\mathrm{II}^2(W))+\bar\eta(V,[F_{n+1},\mathrm{II}(W)]\\-\rho^{-1}\,\mathrm{d}\rho(\mathrm{II}(W))F_{n+1})-\bar\eta(\mathrm{II}(V),[F_{n+1},W]-\rho^{-1}\,\mathrm{d}\rho(W)F_{n+1})\\=-\rho^{-1}(\nabla^{\bar g}_{W}\,\mathrm{d}\rho)(V)-\bar\eta(V,\mathrm{II}^2(W))+\bar\eta(V,[F_{n+1},\mathrm{II}(W)])-\bar\eta(\mathrm{II}(V),[F_{n+1},W]).$$

On the other hand, from the Koszul formula for $\nabla^{\bar\eta}$, we know that

$$2\bar\eta(V,\nabla^{\bar\eta}_{F_{n+1}}\{\mathrm{II}(W)\})\\=F_{n+1}\{\bar\eta(V,\mathrm{II}(W))\}+\bar\eta(V,[F_{n+1},\mathrm{II}(W)])-\bar\eta(\mathrm{II}(W),[F_{n+1},V])\\=2F_{n+1}\{\bar\eta(V,\mathrm{II}(W))\}+2\bar\eta(\nabla^{\bar\eta}_{F_{n+1}}V,\nabla^{\bar\eta}_{W}F_{n+1}),$$

which gives

$$\bar\eta(V,[F_{n+1},\mathrm{II}(W)])-\bar\eta(\mathrm{II}(W),[F_{n+1},V])\\=F_{n+1}\{\bar\eta(V,\mathrm{II}(W))\}+2\bar\eta(\nabla^{\bar\eta}_{F_{n+1}}V,\nabla^{\bar\eta}_{W}F_{n+1}).$$

Then, the equation above for $\bar\eta(V,R_{\bar\eta}(W,F_{n+1})(F_{n+1}))$ becomes

$$\bar\eta(V,R_{\bar\eta}(W,F_{n+1})(F_{n+1}))\\=-\rho^{-1}(\nabla^{\bar g}_{W}\,\mathrm{d}\rho)(V)-\bar\eta(V,\mathrm{II}^2(W))+F_{n+1}\{\bar\eta(V,\mathrm{II}(W))\}+2\bar\eta(\nabla^{\bar\eta}_{F_{n+1}}V,\nabla^{\bar\eta}_{W}F_{n+1})\\+\bar\eta(\mathrm{II}(W),[F_{n+1},V])-\bar\eta(\mathrm{II}(V),[F_{n+1},W])=F_{n+1}\{\mathrm{II}(V,W))\}\\-\rho^{-1}(\nabla^{\bar g}_{W}\,\mathrm{d}\rho)(V)-\bar\eta(V,\mathrm{II}^2(W))-2\bar\eta(\rho^{-1}\nabla^{\eta}_{E_{n+1}}V+\rho^{-1}\Theta(V)-\mathrm{II}(V)\\-\rho^{-1}[\zeta,V],\mathrm{II}(W))+\bar\eta(\rho^{-1}\nabla^{\eta}_{E_{n+1}}V+\rho^{-1}\Theta(V)\\-\rho^{-1}[\zeta,V],\mathrm{II}(W))-\bar\eta(\rho^{-1}\nabla^{\eta}_{E_{n+1}}W+\rho^{-1}\Theta(W)-\rho^{-1}[\zeta,W],\mathrm{II}(V)).$$

Rewriting yields

$$\begin{aligned} E_{n+1}\{\mathrm{II}(V,W)\} &= \rho F_{n+1}\{\mathrm{II}(V,W)\} + \zeta\{\mathrm{II}(V,W)\} \\ &= \rho\cdot\bar{\eta}(V, R_{\bar{\eta}}(W,F_{n+1})(F_{n+1})) + (\nabla^{\bar{g}}_W \,\mathrm{d}\rho)(V) - \rho\bar{\eta}(V,\mathrm{II}^2(W)) \\ &\quad -\bar{\eta}(\mathrm{II}(V),[\zeta,W]) - \bar{\eta}(\mathrm{II}(W),[\zeta,V]) + \bar{\eta}(\mathrm{II}(V),\nabla^{\eta}_{E_{n+1}}W) \\ &\quad +\bar{\eta}(\mathrm{II}(W),\nabla^{\eta}_{E_{n+1}}V) + \bar{\eta}(\mathrm{II}(V),\Theta(W)) + \bar{\eta}(\mathrm{II}(W),\Theta(V)) \\ &\quad +\zeta\{\mathrm{II}(V,W)\}. \end{aligned}$$

With the help of the equation (ii) in Lemma 3.1 and the identities

$$E_{n+1}\{\mathrm{II}(V,W)\} = (\nabla^{\eta}_{E_{n+1}}\mathrm{II})(V,W) + \mathrm{II}(\nabla^{\eta}_{E_{n+1}}V,W) + \mathrm{II}(V,\nabla^{\eta}_{E_{n+1}}W),$$
$$\zeta\{\mathrm{II}(V,W)\} = (\nabla^{\bar{g}}_{\zeta}\mathrm{II})(V,W) + \mathrm{II}(\nabla^{\bar{g}}_{\zeta}V,W) + \mathrm{II}(V,\nabla^{\bar{g}}_{\zeta}W),$$

we obtain the asserted formula of the proposition. □

Remark 3.1. Contracting the equation in Proposition 3.2 and applying (iii) in Lemma 3.1, we obtain the following formula for $\mathrm{Ric}_{\bar{\eta}}(F_{n+1},F_{n+1})$:

$$\begin{aligned} \rho\cdot\mathrm{Ric}_{\bar{\eta}}(F_{n+1},F_{n+1}) &= \mathrm{Tr}_{\bar{g}}(\nabla^{\eta}_{E_{n+1}}\mathrm{II}) - \mathrm{Tr}_{\bar{g}}(\nabla^{\bar{g}}_{\zeta}\mathrm{II}) + \rho\cdot\mathrm{Tr}_{\bar{g}}(\mathrm{II}^2) \\ &\quad - 2\sum_{i=1}^{n}\bar{\eta}(\mathrm{II}(F_i),\nabla^{\bar{g}}_{F_i}\zeta) - 2\sum_{i=1}^{n}\bar{\eta}(\Theta(F_i),\mathrm{II}(F_i)) \\ &\quad - \sum_{i=1}^{n}(\nabla^{\bar{g}}_{F_i}\,\mathrm{d}\rho)(F_i). \end{aligned}$$

4. Representation of the Dirac equation on manifolds with codimension one foliation

In this section we will represent the Dirac equation on $(Q^{n+1},\bar{\eta})$ hyperbolically with respect to codimension one foliation (see Proposition 4.1 and Corollary 4.1). Let us fix a slice $(M^n,\bar{g})$ of the foliated manifold $(Q^{n+1},\bar{\eta})$. We will identify $\Sigma(Q)_{\bar{\eta}}$ with $\Sigma(Q)_{\eta}$ and $\psi\in\Gamma(\Sigma(Q)_{\bar{\eta}})$ with its pullback $\tilde{K}(\psi)$, via the natural isomorphism $\tilde{K}:\Sigma(Q)_{\bar{\eta}}\to\Sigma(Q)_{\eta}$, and write simply as $\Sigma(Q)$ and ψ, respectively. Depending on the dimension $n+1$ of the manifold Q^{n+1}, we will use two different Clifford multiplications in the subbundle $\Sigma(M)\subset\Sigma(Q)$. For the realization of the Clifford algebra over $\mathbb{R}$, we refer to [11]. Let $\mathrm{Cl}(M)$ (resp. $\mathrm{Cl}(Q)$) denote the Clifford bundle over M^n (resp. Q^{n+1}):

(i) In case of $n=2m$, we use the Clifford multiplication $\mathrm{Cl}(M)\times\Sigma(M)\to\Sigma(M)$ that is naturally related to the one $\mathrm{Cl}(Q)\times\Sigma(Q)\to\Sigma(Q)$ via

$$\pi_*(F_i\cdot\psi) = F_i\cdot(\pi_*\psi),\quad 1\le i\le 2m,$$
$$\pi_*(F_{2m+1}\cdot\psi) = (\sqrt{-1})^{m+1}\mu_{\bar{g}}\cdot(\pi_*\psi),$$

where $\pi_* : \Sigma(Q) \to \Sigma(M)$ is the restriction map and $\mu_{\bar{g}}$ is the volume element of $(M^{2m}, \bar{g})$. The second relation is an immediate consequence of the algebraic relation

$$F_{2m+1} \cdot \psi = (\sqrt{-1})^{m+1} F_1 \cdots F_{2m} \cdot \psi.$$

(ii) In the other case $n = 2m-1$, we identify the spinor bundle $\Sigma(M)$ with the positive part $\Sigma^+(Q)$ of the bundle $\Sigma(Q)$ restricted to M^{2m-1}, and we use the Clifford multiplication $\mathrm{Cl}(M) \times \Sigma(M) \to \Sigma(M)$ that is naturally related to $\mathrm{Cl}^+(Q) \times \Sigma^+(Q) \to \Sigma^+(Q)$ via

$$\pi_*^+(F_i \cdot F_{2m} \cdot \psi^+) = F_i \cdot (\pi_*^+ \psi^+), \quad 1 \le i \le 2m-1, \qquad \pi_*^+(F_k \cdot F_l \cdot \psi^+) = F_k \cdot F_l \cdot (\pi_*^+ \psi^+), \quad 1 \le k < l \le 2m-1,$$

where $\pi_*^+ : \Sigma^+(Q) \to \Sigma(M)$ is the restriction map and $\mathrm{Cl}^+(Q)$ is the positive part of $\mathrm{Cl}(Q)$.

Recall that the spin derivatives $\nabla^{\bar{\eta}}\psi$, $\nabla^{\bar{g}}\psi$ are related, on Q^{n+1}, by

$$\nabla^{\bar{\eta}}_V \psi = \nabla^{\bar{g}}_V \psi + \tfrac{1}{2}\mathrm{II}(V) \cdot F_{n+1} \cdot \psi.$$

In view of the rule of Clifford multiplication described above, we find that, in case of $n = 2m$, the formula is projected to the slice $(M^{2m}, \bar{g})$ as

$$\pi_*(\nabla^{\bar{\eta}}_V \psi) = \nabla^{\bar{g}}_V (\pi_* \psi) + \tfrac{1}{2}(\sqrt{-1})^{m+1}\mathrm{II}(V) \cdot \mu_{\bar{g}} \cdot (\pi_* \psi).$$

However, in the other case $n = 2m-1$, the projection is only possible if $\psi = \psi^+$ belongs entirely to the positive part $\Sigma^+(Q)$ of $\Sigma(Q)$, the projected formula being given by

$$\pi_*^+(\nabla^{\bar{\eta}}_V \psi^+) = \nabla^{\bar{g}}_V (\pi_*^+ \psi^+) + \tfrac{1}{2}\mathrm{II}(V) \cdot (\pi_*^+ \psi^+).$$

Nevertheless, we may regard not only $\nabla^{\bar{g}}_V \psi^+$ but also $\nabla^{\bar{g}}_V \psi^-$, $\psi^- \in \Gamma(\Sigma^-(Q))$ (e.g. $\psi^- = F_{2m} \cdot \psi^+$), as well-defined spinor fields on Q^{2m}, not projected to the slice M^{2m-1}. Therefore, the following formula makes sense.

Lemma 4.1.

$$\nabla^{\bar{g}}_V (F_{2m} \cdot \psi^+) = F_{2m} \cdot \nabla^{\bar{g}}_V \psi^+.$$

Proof. We compute

$$\begin{aligned}\nabla^{\bar{\eta}}_V (F_{2m} \cdot \psi^+) &= \nabla^{\bar{\eta}}_V F_{2m} \cdot \psi^+ + F_{2m} \cdot \nabla^{\bar{\eta}}_V \psi^+ \\ &= -\mathrm{II}(V) \cdot \psi^+ + F_{2m} \cdot \{\nabla^{\bar{g}}_V \psi^+ + \tfrac{1}{2}\mathrm{II}(V) \cdot F_{2m} \cdot \psi^+\} \\ &= F_{2m} \cdot \nabla^{\bar{g}}_V \psi^+ - \tfrac{1}{2}\mathrm{II}(V) \cdot \psi^+.\end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla^{\bar{\eta}}_V (F_{2m} \cdot \psi^+) &= \nabla^{\bar{g}}_V (F_{2m} \cdot \psi^+) + \tfrac{1}{2}\mathrm{II}(V) \cdot F_{2m} \cdot F_{2m} \cdot \psi^+ \\ &= \nabla^{\bar{g}}_V (F_{2m} \cdot \psi^+) - \tfrac{1}{2}\mathrm{II}(V) \cdot \psi^+.\end{aligned}$$

Comparing the latter equation with the former, we complete the proof. □

In order to represent the Dirac equation

$$D_{\bar{\eta}}\psi = \lambda_Q\psi = \sum_{i=1}^{n} F_i\cdot\nabla^{\bar{g}}_{F_i}\psi - \tfrac{1}{2}(\mathrm{Tr}_{\bar{g}}\mathrm{II})F_{n+1}\cdot\psi + F_{n+1}\cdot\nabla^{\bar{\eta}}_{F_{n+1}}\psi,$$

with respect to reference metric, we need the following lemma that one verifies straightforwardly using Proposition 3.1.

Lemma 4.2.

$$\begin{aligned}
&\rho\cdot\{\tilde{K}\circ\nabla^{\bar{\eta}}_{F_{n+1}}\circ(\tilde{K})^{-1}\}(\varphi)\\
&= \nabla^{\eta}_{E_{n+1}}\varphi - \nabla^{g}_{\zeta}\varphi - \frac{\rho}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\mathrm{II}\circ L^{-1})(E_i)\cdot\varphi - \frac{1}{4}\sum_{i=1}^{n}E_i\cdot(\nabla^{\eta}_{E_{n+1}}L)\\
&\quad\times(L^{-1}E_i)\cdot\varphi + \frac{1}{4}\sum_{i=1}^{n}E_i\cdot(\nabla^{g}_{\zeta}L)(L^{-1}E_i)\cdot\varphi + \frac{1}{4}\sum_{i=1}^{n}E_i\cdot L(\nabla^{g}_{L^{-1}E_i}\zeta)\cdot\varphi\\
&\quad+\frac{1}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\Theta\circ L^{-1})(E_i)\cdot\varphi + \frac{1}{2}\sum_{i=1}^{n}\mathrm{d}\rho(L^{-1}E_i)E_i\cdot E_{n+1}\cdot\varphi.
\end{aligned}$$

Lemma 4.2, combined with Proposition 2.3, yields the following hyperbolic representation of the Dirac equation immediately.

Proposition 4.1.

$$\begin{aligned}
\nabla^{\eta}_{E_{n+1}}\psi &= \nabla^{g}_{\zeta}\psi - \lambda_Q\rho E_{n+1}\cdot\psi + \frac{\rho}{2}(\mathrm{Tr}_{\bar{g}}\mathrm{II})\psi\\
&\quad+\rho E_{n+1}\cdot\left\{\sum_{i=1}^{n}E_i\cdot\nabla^{g}_{L^{-1}E_i}\psi + \frac{1}{4}\sum_{j,k,l=1}^{n}(\Lambda_g)_{jkl}E_j\cdot E_k\cdot E_l\cdot\psi\right\}\\
&\quad+\frac{1}{4}\sum_{i=1}^{n}E_i\cdot(\nabla^{\eta}_{E_{n+1}}L)(L^{-1}E_i)\cdot\psi + \frac{\rho}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\mathrm{II}\circ L^{-1})(E_i)\cdot\psi\\
&\quad-\frac{1}{4}\sum_{i=1}^{n}E_i\cdot(\nabla^{g}_{\zeta}L)(L^{-1}E_i)\cdot\psi - \frac{1}{4}\sum_{i=1}^{n}E_i\cdot L(\nabla^{g}_{L^{-1}E_i}\zeta)\cdot\psi\\
&\quad-\frac{1}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\Theta\circ L^{-1})(E_i)\cdot\psi - \frac{1}{2}\sum_{i=1}^{n}d\rho(L^{-1}E_i)E_i\cdot E_{n+1}\cdot\psi.
\end{aligned}$$

Although the equation in Proposition 4.1 is valid in both cases, $n = 2m$ and $n = 2m-1$, it is also very useful, in the latter case $n = 2m-1$, to consider the decomposition of spinor fields

$$\psi = \psi^+ + F_{2m}\cdot\varphi^+, \qquad \psi^+,\varphi^+ \in \Gamma(\Sigma^+(Q)),$$

and rewrite the representation in Proposition 4.1 equivalently as follows.

Corollary 4.1.

$$\begin{aligned}\nabla^{\eta}_{E_{2m}}\psi^{+} &= \nabla^{g}_{\zeta}\psi^{+} + \lambda_Q\rho\varphi^{+} + \frac{\rho}{2}(\mathrm{Tr}_{\bar g}\mathrm{II})\psi^{+}\\ &\quad + \rho E_{2m}\cdot\left\{\sum_{i=1}^{2m-1} E_i\cdot\nabla^{g}_{L^{-1}E_i}\psi^{+} + \frac{1}{4}\sum_{j,k,l=1}^{2m-1}(\Lambda_g)_{jkl}E_j\cdot E_k\cdot E_l\cdot\psi^{+}\right\}\\ &\quad + \frac{1}{4}\sum_{i=1}^{2m-1} E_i\cdot(\nabla^{\eta}_{E_{2m}}L)(L^{-1}E_i)\cdot\psi^{+} + \frac{\rho}{4}\sum_{i=1}^{2m-1}E_i\cdot(L\circ\mathrm{II}\circ L^{-1})\\ &\quad \times(E_i)\cdot\psi^{+} - \frac{1}{4}\sum_{i=1}^{2m-1}E_i\cdot(\nabla^{g}_{\zeta}L)(L^{-1}E_i)\cdot\psi^{+}\\ &\quad - \frac{1}{4}\sum_{i=1}^{2m-1}E_i\cdot L(\nabla^{g}_{L^{-1}E_i}\zeta)\cdot\psi^{+} - \frac{1}{4}\sum_{i=1}^{2m-1}E_i\cdot(L\circ\Theta\circ L^{-1})(E_i)\cdot\psi^{+}\\ &\quad - \frac{1}{2}\sum_{i=1}^{2m-1}\mathrm{d}\rho(L^{-1}E_i)E_i\cdot E_{2m}\cdot\psi^{+},\end{aligned}$$

$$\begin{aligned}\nabla^{\eta}_{E_{2m}}\varphi^{+} &= \nabla^{g}_{\zeta}\varphi^{+} - \lambda_Q\rho\psi^{+} + \frac{\rho}{2}(\mathrm{Tr}_{\bar g}\mathrm{II})\varphi^{+}\\ &\quad - \rho E_{2m}\cdot\left\{\sum_{i=1}^{2m-1} E_i\cdot\nabla^{g}_{L^{-1}E_i}\varphi^{+} + \frac{1}{4}\sum_{j,k,l=1}^{2m-1}(\Lambda_g)_{jkl}E_j\cdot E_k\cdot E_l\cdot\varphi^{+}\right\}\\ &\quad + \frac{1}{4}\sum_{i=1}^{2m-1} E_i\cdot(\nabla^{\eta}_{E_{2m}}L)(L^{-1}E_i)\cdot\varphi^{+} + \frac{\rho}{4}\sum_{i=1}^{2m-1}E_i\cdot(L\circ\mathrm{II}\circ L^{-1})\\ &\quad \times(E_i)\cdot\varphi^{+} - \frac{1}{4}\sum_{i=1}^{2m-1}E_i\cdot(\nabla^{g}_{\zeta}L)(L^{-1}E_i)\cdot\varphi^{+}\\ &\quad - \frac{1}{4}\sum_{i=1}^{2m-1}E_i\cdot L(\nabla^{g}_{L^{-1}E_i}\zeta)\cdot\varphi^{+} - \frac{1}{4}\sum_{i=1}^{2m-1}E_i\cdot(L\circ\Theta\circ L^{-1})(E_i)\cdot\varphi^{+}\\ &\quad + \frac{1}{2}\sum_{i=1}^{2m-1}\mathrm{d}\rho(L^{-1}E_i)E_i\cdot E_{2m}\cdot\varphi^{+}.\end{aligned}$$

We close this section with representing the energy-momentum tensor

$$T_{\bar\eta}(X,Y) = \tfrac{1}{4}\epsilon(X\cdot\nabla^{\bar\eta}_{Y}\psi + Y\cdot\nabla^{\bar\eta}_{X}\psi,\psi),\qquad \epsilon=\pm1,$$

hyperbolically with respect to codimension one foliation. To this end, it is important to notice that, if ψ be a solution of the Dirac equation $D_{\bar\eta}\psi = \lambda_Q\psi$, $\lambda_Q\in\mathbb{R}$, then the following equation is valid:

$$\nabla^{\bar{\eta}}_{F_{n+1}}\psi = -\lambda_Q F_{n+1}\cdot\psi + \frac{1}{2}(\mathrm{Tr}_{\bar{g}}\mathrm{II})\psi + F_{n+1}\cdot\left(\sum_{i=1}^{n} F_i\cdot\nabla^{\bar{g}}_{F_i}\psi\right).$$

Proposition 4.2. *For any solution ψ of the Dirac equation $D_{\bar{\eta}}\psi = \lambda_Q\psi$ on $(Q^{n+1},\bar{\eta})$, we have*

$$\mathrm{Tr}_{\bar{\eta}}(T_{\bar{\eta}}) = \tfrac{1}{2}\epsilon\lambda_Q(\psi,\psi),$$

$$T_{\bar{\eta}}(V,W)=\tfrac{1}{4}\epsilon(V\cdot\nabla^{\bar{g}}_W\psi+W\cdot\nabla^{\bar{g}}_V\psi,\psi) + \tfrac{1}{8}\epsilon(\{V\cdot\mathrm{II}(W)+W\cdot\mathrm{II}(V)\}\cdot F_{n+1}\cdot\psi,\psi),$$

$$T_{\bar{\eta}}(V,F_{n+1}) = \frac{\epsilon}{4}\left(F_{n+1}\cdot\left\{\nabla^{\bar{g}}_V\psi - V\cdot\left(\sum_{i=1}^{n}F_i\cdot\nabla^{\bar{g}}_{F_i}\psi\right)\right\},\psi\right),$$

$$T_{\bar{\eta}}(F_{n+1},F_{n+1}) = -\frac{\epsilon}{2}\left(\left(\sum_{i=1}^{n}F_i\cdot\nabla^{\bar{g}}_{F_i}\psi\right)-\lambda_Q\psi,\psi\right).$$

In case of $n = 2m-1$, we consider the decomposition $\psi = \psi^+ + F_{2m}\cdot\varphi^+$ and can equivalently rewrite the formulas in Proposition 4.2 as follows.

Corollary 4.2. *For any solution ψ of the Dirac equation $D_{\bar{\eta}}\psi = \lambda_Q\psi$ on $(Q^{2m},\bar{\eta})$, where $\psi = \psi^+ + F_{2m}\cdot\varphi^+$, we have*

$$\mathrm{Tr}_{\bar{\eta}}(T_{\bar{\eta}}) = \tfrac{1}{2}\epsilon\lambda_Q\{(\psi^+,\psi^+)+(\varphi^+,\varphi^+)\},$$

$$\begin{aligned}T_{\bar{\eta}}(V,W) = {}&\tfrac{1}{4}\epsilon(V\cdot\nabla^{\bar{g}}_W\psi^+ + W\cdot\nabla^{\bar{g}}_V\psi^+, F_{2m}\cdot\varphi^+) + \tfrac{1}{4}\epsilon(V\cdot\nabla^{\bar{g}}_W\varphi^+ + W\cdot\nabla^{\bar{g}}_V\varphi^+,\\ &F_{2m}\cdot\psi^+) + \tfrac{1}{4}\epsilon(\{V\cdot\mathrm{II}(W)+W\cdot\mathrm{II}(V)\}\cdot\psi^+,\varphi^+)+\tfrac{1}{2}\epsilon\mathrm{II}(V,W)(\psi^+,\varphi^+),\end{aligned}$$

$$\begin{aligned}T_{\bar{\eta}}(V,F_{2m}) = {}&\frac{\epsilon}{4}\left(\nabla^{\bar{g}}_V\psi^+ - V\cdot\left(\sum_{i=1}^{2m-1}F_i\cdot\nabla^{\bar{g}}_{F_i}\psi^+\right),\varphi^+\right)\\ &-\frac{\epsilon}{4}\left(\nabla^{\bar{g}}_V\varphi^+ - V\cdot\left(\sum_{i=1}^{2m-1}F_i\cdot\nabla^{\bar{g}}_{F_i}\varphi^+\right),\psi^+\right),\end{aligned}$$

$$\begin{aligned}T_{\bar{\eta}}(F_{2m},F_{2m}) = {}&\frac{\epsilon}{2}\left(F_{2m}\cdot\left(\sum_{i=1}^{2m-1}F_i\cdot\nabla^{\bar{g}}_{F_i}\psi^+\right),\varphi^+\right)\\ &+\frac{\epsilon}{2}\left(F_{2m}\cdot\left(\sum_{i=1}^{2m-1}F_i\cdot\nabla^{\bar{g}}_{F_i}\varphi^+\right),\psi^+\right)\\ &+\frac{\epsilon\lambda_Q}{2}\{(\psi^+,\psi^+)+(\varphi^+,\varphi^+)\}.\end{aligned}$$

5. A sufficient condition for the existence of solutions to the weak Killing equation

Let us suppose that $(Q^{n+1}, \bar{\eta})$ satisfies

$$\mathrm{Ric}_{\bar{\eta}}(V, W) = \tfrac{1}{2}(S_{\bar{\eta}})\bar{\eta}(V, W), \qquad \mathrm{Ric}_{\bar{\eta}}(V, F_{n+1}) = 0, \qquad \mathrm{d}S_{\bar{\eta}}(V) = 0$$

for all $V, W \in E_{n+1}^{\perp}$. Then the weak Killing equation becomes

$$\nabla_V^{\bar{\eta}}\psi = \nabla_V^{\bar{g}}\psi + \frac{1}{2}\mathrm{II}(V)\cdot F_{n+1}\cdot\psi = \frac{\mathrm{d}S_{\bar{\eta}}(F_{n+1})}{2nS_{\bar{\eta}}}V\cdot F_{n+1}\cdot\psi,$$

and

$$\begin{aligned}\nabla_{F_{n+1}}^{\bar{\eta}}\psi &= -\lambda_Q F_{n+1}\cdot\psi + \frac{(n+1)\,\mathrm{d}S_{\bar{\eta}}(F_{n+1})}{2nS_{\bar{\eta}}}\psi + \frac{\mathrm{d}S_{\bar{\eta}}(F_{n+1})}{2nS_{\bar{\eta}}}F_{n+1}\cdot F_{n+1}\cdot\psi\\ &= -\lambda_Q F_{n+1}\cdot\psi + \frac{\mathrm{d}S_{\bar{\eta}}(F_{n+1})}{2S_{\bar{\eta}}}\psi.\end{aligned}$$

Thus we have proved the following proposition.

Proposition 5.1. *Let $(Q^{n+1}, \bar{\eta})$ satisfy the following conditions:*

$$\mathrm{Ric}_{\bar{\eta}}(V, W) = \frac{S_{\bar{\eta}}}{2}\bar{\eta}(V, W), \qquad \mathrm{Ric}_{\bar{\eta}}(V, F_{n+1}) = 0, \qquad \mathrm{d}S_{\bar{\eta}}(V) = 0,$$
$$\mathrm{II}(V, W) = \frac{\mathrm{d}S_{\bar{\eta}}(F_{n+1})}{nS_{\bar{\eta}}}\bar{\eta}(V, W).$$

Under this assumption the weak Killing equation is equivalent to the system of differential equations

$$\nabla_V^{\bar{g}}\psi = 0 \quad \text{and} \quad \nabla_{F_{n+1}}^{\bar{\eta}}\psi = -\lambda_Q F_{n+1}\cdot\psi + \tfrac{1}{2}\mathrm{Tr}_{\bar{g}}(\mathrm{II})\psi.$$

As an application of Proposition 5.1, we are going to prove below that every parallel spinor may evolve to a WK-spinor (Theorem 5.1). For this purpose, we first show that there indeed exist some special metrics satisfying the hypothesis of Proposition 5.1. Let $Q^{n+1} = M^n \times \mathbb{R}$ be a product manifold, and let the product metric $\eta = g_M \times g_{\mathbb{R}}$ be the reference metric on Q^{n+1}, where g_M indicates an arbitrary Riemannian metric on M^n and $g_{\mathbb{R}}$ the standard metric on the real line $\mathbb{R}$. We write $g_{\mathbb{R}} = \mathrm{d}t \otimes \mathrm{d}t$, using the standard coordinate $t \in \mathbb{R}$. By $(E_1, \ldots, E_n)$ we denote a local orthonormal frame on (M^n, g_M) as well as its lift to (Q^{n+1}, η). Let $E_{n+1} = d/\mathrm{d}t$ denote the unit vector field on $(\mathbb{R}, g_{\mathbb{R}})$ as well as the lift to (Q^{n+1}, η). Then it is clear that $\Theta(V) = -\nabla_V^{\eta}E_{n+1} = 0$ for all vector fields $V \in E_{n+1}^{\perp}$. For simplicity, we denoted by $\mathrm{WP}(g_M; a)$ the following class of metrics (the *warped products of g_M and $g_{\mathbb{R}}$*):

$$\bar{\eta} = \mathrm{e}^{f}\left(\sum_{i=1}^{n} E^i \otimes E^i\right) + \mathrm{e}^{af}\,\mathrm{d}t \otimes \mathrm{d}t,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function and $a \in \mathbb{R}$ is a real number.

Lemma 5.1. *For all $\bar\eta \in \mathrm{WP}(g_M; a)$, we have:*

$$\mathrm{II}(V,W) = -\frac{\mathrm{e}^{-((a/2)-1)f} f_t}{2}\eta(V,W), \qquad \bar\eta(V,\mathrm{II}^2(W)) = \frac{\mathrm{e}^{-(a-1)f} f_t f_t}{4}\eta(V,W),$$
$$\mathrm{Tr}_{\bar g}(\mathrm{II}) = -\frac{n\,\mathrm{e}^{-(a/2)f} f_t}{2}, \qquad \mathrm{Tr}_{\bar g}(\mathrm{II}^2) = \frac{n\,\mathrm{e}^{-af} f_t f_t}{4},$$
$$(\nabla^{\eta}_{E_{n+1}}\mathrm{II})(V,W) = \left\{-\frac{\mathrm{e}^{-((a/2)-1)f} f_{tt}}{2} + \frac{(a-2)\,\mathrm{e}^{-((a/2)-1)f} f_t f_t}{4}\right\}\eta(V,W),$$

where we have used the shorthand notation $f_t := \mathrm{d}f(E_{n+1})$ and $f_{tt} := \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = (\nabla^{\eta}_{E_{n+1}}\mathrm{d}f)(E_{n+1})$.

Proof. Since $L = \mathrm{e}^{f/2} I$ (in the notations of Section 3), we have

$$(\nabla^g_V L)(W) = 0 \quad \text{and} \quad (\nabla^{\eta}_{E_{n+1}} L)(W) = \frac{\mathrm{e}^{(f/2)} f_t}{2} W,$$

and hence Corollary 3.1 gives

$$\mathrm{II}(V,W) = -\frac{\mathrm{e}^{-((a/2)-1)f} f_t}{2}\eta(V,W).$$

Using this, one checks all the equations of the lemma easily. □

Substituting Lemma 5.1 in Proposition 3.2 as well as in the Gauss–Codazzi equations (Lemma 3.1), we obtain the following lemma immediately.

Lemma 5.2. *For all $\bar\eta \in \mathrm{WP}(g_M; a)$, we have:*

$$\mathrm{Ric}_{\bar\eta}(V,W) = \mathrm{Ric}_g(V,W) + \left\{-\frac{f_{tt}}{2\,\mathrm{e}^{(a-1)f}} + \frac{(a-n) f_t f_t}{4\,\mathrm{e}^{(a-1)f}}\right\}\eta(V,W),$$
$$\mathrm{Ric}_{\bar\eta}(V,F_{n+1}) = 0, \qquad \mathrm{Ric}_{\bar\eta}(F_{n+1},F_{n+1}) = -\frac{n f_{tt}}{2\,\mathrm{e}^{af}} + \frac{n(a-1) f_t f_t}{4\,\mathrm{e}^{af}},$$
$$S_{\bar\eta} = \mathrm{e}^{-f} S_g - \frac{n f_{tt}}{\mathrm{e}^{af}} + \frac{n(2a-n-1) f_t f_t}{4\,\mathrm{e}^{af}}.$$

Lemma 5.3. *$\bar\eta \in \mathrm{WP}(g_M; a)$ satisfies the hypothesis of Proposition 5.1 if and only if either*

$$f(t) = ct \quad \left(a = \frac{n}{2}\right)$$

or

$$f(t) = \frac{4}{n-2a}\log\left\{1 + \frac{(n-2a)ct}{4}\right\} \quad \left(a \neq \frac{n}{2},\ 1 + \frac{(n-2a)ct}{4} > 0\right),$$

where $f(0) = 0$ and $c := f_t(0)$. The scalar curvature of the metric $\bar\eta$ is non-positive and is equal to

$$-\frac{nc^2}{4}\mathrm{e}^{-(nc/2)t} \quad \text{and} \quad -\frac{nc^2}{4}\left\{\frac{4+(n-2a)ct}{4}\right\}^{-2n/(n-2a)},$$

respectively.

Proof. Using Lemmas 5.1 and 5.2, we compute

$$\mathrm{Ric}_{\bar\eta}(V,W)-\frac{1}{2}S_{\bar\eta}\bar\eta(V,W)=\frac{n-1}{8}\mathrm{e}^{-af}\{4f_{tt}+(n-2a)f_t f_t\}\bar\eta(V,W),$$

and

$$\begin{aligned}&nS_{\bar\eta}\mathrm{II}(V,W)-\mathrm{d}S_{\bar\eta}(F_{n+1})\bar\eta(V,W)\\&\quad=\frac{n}{8}\mathrm{e}^{-3/2af}\{8f_{ttt}-4(4a-2n-1)f_t f_{tt}+(2a-n)(2a-n-1)f_t f_t f_t\}\bar\eta(V,W).\end{aligned}$$

Since $4f_{tt}+(n-2a)f_t f_t=0$ implies

$$8f_{ttt}-4(4a-2n-1)f_t f_{tt}+(2a-n)(2a-n-1)f_t f_t f_t=0,$$

the hypothesis of Proposition 5.1 is satisfied if and only if $4f_{tt}+(n-2a)f_t f_t=0$, which can be solved completely as given in the lemma. □

Proposition 5.2. *Let (M^{2m},g_M) be a Riemannian manifold admitting parallel spinors. Then, for any real number $\lambda_Q\in\mathbb{R}$, there exists a warped product metric $\bar\eta\in\mathrm{WP}(g_M;a)$* (see Lemma 5.3) *on $Q^{2m+1}=M^{2m}\times\mathbb{R}$ such that $(Q^{2m+1},\bar\eta)$ admits a WK-spinor to WK-number λ_Q.*

Proof. Let $\psi_M^+\in\Gamma(\Sigma^+(M))$ be a parallel spinor on M^{2m}. Let $\psi^+=h^+\psi_M^+$ be a spinor field on Q^{2m+1} defined by

$$\psi^+(x,t)=h^+(t)\psi_M^+(x),\quad (x,t)\in M^{2m}\times\mathbb{R},$$

where $h^+:\mathbb{R}\to\mathbb{C}$ is a complex-valued function with $h^+(0)=1$. Let $\bar\eta\in\mathrm{WP}(g_M;a)$ be a warped product metric given as in Lemma 5.3. Now recall that, in our realization of Clifford algebra, the volume form μ_M of (M^{2m},g_M) acts on $\Sigma^\pm(M)$ via

$$\mu_M\cdot\psi_M^+=(\sqrt{-1})^m\psi_M^+,\qquad \mu_M\cdot\psi_M^-=-(\sqrt{-1})^m\psi_M^-.$$

Since $E_{2m+1}\cdot\varphi=(\sqrt{-1})^{m+1}\mu_M\cdot\varphi$ for all spinor fields φ on Q^{2m+1}, we have

$$E_{2m+1}\cdot\psi_M^+=(\sqrt{-1})^{2m+1}\psi_M^+,\qquad E_{2m+1}\cdot\psi_M^-=-(\sqrt{-1})^{2m+1}\psi_M^-.$$

By Proposition 5.1, ψ^+ is a WK-spinor to WK-number λ_Q if and only if the function h^+ satisfies

$$h_t^+=-(\sqrt{-1})^{2m+1}\lambda_Q\,\mathrm{e}^{(a/2)f}h^+-\tfrac{1}{2}(m)f_t h^+,$$

which obviously allows a global solution. □

We now extend Proposition 5.2 so as to include the case $n=2m-1$.

Theorem 5.1. *Let (M^n,g_M) be a Riemannian manifold admitting parallel spinors. Then, for any real number $\lambda_Q\in\mathbb{R}$, there exists a warped product metric $\bar\eta\in\mathrm{WP}(g_M;a)$* (see Lemma 5.3) *on $Q^{n+1}=M^n\times\mathbb{R}$ such that $(Q^{n+1},\bar\eta)$ admits a WK-spinor to WK-number λ_Q.*

Proof. Because of Proposition 5.2, it suffices to prove the theorem for the case $n = 2m-1$. Let $\varphi_M^+ \in \Gamma(\Sigma^+(Q))$ be a parallel spinor on M^{2m-1}, and let $\varphi = h^+\varphi_M^+ + k^+ E_{2m}\cdot\varphi_M^+$ be a spinor field on Q^{2m} defined by

$$\varphi(x,t) = h^+(t)\varphi_M^+(x) + k^+(t)E_{2m}\cdot\varphi_M^+(x), \quad (x,t)\in M^{2m-1}\times\mathbb{R},$$

where $h^+, k^+ : \mathbb{R}\to\mathbb{C}$ are complex-valued functions with $h^+(0) = k^+(0) = 1$. Let $\bar\eta \in \mathrm{WP}(g_M; a)$ be a warped product metric given as in Lemma 5.3. By Proposition 5.1, φ is a WK-spinor to WK-number λ_Q if and only if (h^+, k^+) satisfies the system of differential equations,

$$h_t^+ = -\frac{2m-1}{4} f_t h^+ + \lambda_Q\, \mathrm{e}^{(a/2)f} k^+ \quad \text{and} \quad k_t^+ = -\lambda_Q\, \mathrm{e}^{(a/2)f} h^+ - \frac{2m-1}{4} f_t k^+.$$

This is a linear homogeneous system and hence allows a global solution. □

Remark 5.1. The WK-spinors constructed on $\mathbb{R}^3$ at the end of Section 8 in the paper [11] (see p. 171) are a special case of Theorem 5.1 (for the metric $\bar\eta \in \mathrm{WP}(g_M; a)$ with $f(t) = ct$).

6. The initial-value formulation for the Einstein–Dirac equation

In this section we set up an invariant initial-value formulation for the Einstein–Dirac equation

$$\mathrm{Ric}_{\bar\eta} - \tfrac{1}{2} S_{\bar\eta}\bar\eta = T_{\bar\eta}, \qquad D_{\bar\eta}\psi = \lambda_Q\psi, \quad \lambda_Q\in\mathbb{R},$$

where

$$T_{\bar\eta}(X,Y) = \tfrac{1}{4}\epsilon(X\cdot\nabla_Y^{\bar\eta}\psi + Y\cdot\nabla_X^{\bar\eta}\psi, \psi), \quad \epsilon = \pm 1.$$

The formulation will be applied in the next section to establish a local existence theorem for a specific class of initial data sets (see Theorem 7.1). Following the work [7] as a guideline, we can indeed express the evolution equations as well as the constraints in an invariant form. For simplicity, we write

$$\Delta := \mathrm{Ric}_{\bar\eta} - \tfrac{1}{2} S_{\bar\eta}\bar\eta - T_{\bar\eta}.$$

The tensor field Δ decomposes into three parts

$$\Delta = \Delta^E + \{\Delta^M \otimes F^{n+1} + F^{n+1}\otimes\Delta^M\} + \Delta^H(F^{n+1}\otimes F^{n+1}),$$

where

$$\Delta^E = \sum_{i,j=1}^{n} \Delta(F_i, F_j) F^i\otimes F^j, \qquad \Delta^M = \sum_{i=1}^{n}\Delta(F_{n+1}, F_i)F^i,$$

$$\Delta^H = \Delta(F_{n+1}, F_{n+1}).$$

Restricting the equations, $\Delta^M = 0$ and $\Delta^H = 0$, to a fixed slice, we obtain the *momentum constraint*

$$T_{\bar{g}}(F_{n+1}, V) = \mathrm{d}(\mathrm{Tr}_{\bar{g}}\mathrm{II})(V) - \mathrm{div}_{\bar{g}}(\mathrm{II})(V),$$

and the *Hamiltonian constraint*

$$T_{\bar{g}}(F_{n+1}, F_{n+1}) = -\tfrac{1}{2}S_{\bar{g}} + \tfrac{1}{2}(\mathrm{Tr}_{\bar{g}}\mathrm{II})^2 - \tfrac{1}{2}\mathrm{Tr}_{\bar{g}}(\mathrm{II}^2),$$

where $T_{\bar{g}}$ denotes the restriction of $T_{\bar{\eta}}$ to the slices. The information on the evolution should be contained in

$$\Delta^E = 0,$$

or any combination of it with the constraints. The evolution equations should be chosen in such a way that, under the evolution, local preservation of the constraints is guaranteed. We consider the evolution equations of the form

$$\Delta(V, W) = \Delta(F_{n+1}, F_{n+1}) \cdot \bar{\eta}(V, W) \quad \text{and} \quad D_{\bar{\eta}}\psi = \lambda_Q \psi, \quad \lambda_Q \in \mathbb{R}.$$

Note that, under this evolution, the tensor Δ becomes

$$\Delta = \Delta^M \otimes F^{n+1} + F^{n+1} \otimes \Delta^M + \Delta^H \cdot \bar{\eta}.$$

Proposition 6.1. *The equation*

$$\Delta(V, W) = \Delta(F_{n+1}, F_{n+1}) \cdot \bar{\eta}(V, W),$$

is equivalent to

$$\mathrm{Ric}_{\bar{\eta}}(V, W) = \frac{1}{n-1}\{S_{\bar{g}} + \mathrm{Tr}_{\bar{g}}(\mathrm{II}^2) - (\mathrm{Tr}_{\bar{g}}\mathrm{II})^2 - \mathrm{Tr}_{\bar{\eta}}(T_{\bar{\eta}}) + 2T_{\bar{\eta}}(F_{n+1}, F_{n+1})\} \cdot \bar{\eta}(V, W) + T_{\bar{\eta}}(V, W).$$

Proof. Contracting the equation

$$\Delta(V, W) = \Delta(F_{n+1}, F_{n+1}) \cdot \bar{\eta}(V, W),$$

we see that

$$\Delta(F_{n+1}, F_{n+1}) = -\frac{n-1}{2(n+1)}S_{\bar{\eta}} - \frac{1}{n+1}\mathrm{Tr}_{\bar{\eta}}(T_{\bar{\eta}}),$$

from which it follows that

$$\mathrm{Ric}_{\bar{\eta}}(V, W) = \frac{1}{n+1}\{S_{\bar{\eta}} - \mathrm{Tr}_{\bar{\eta}}(T_{\bar{\eta}})\} \cdot \bar{\eta}(V, W) + T_{\bar{\eta}}(V, W).$$

Let us contract this equation. Then, with the help of the Gauss equation (iii) in Lemma 3.1, we obtain

$$S_{\bar{\eta}} = \frac{n+1}{n-1}\{S_{\bar{g}} + \mathrm{Tr}_{\bar{g}}(\mathrm{II}^2) - (\mathrm{Tr}_{\bar{g}}\mathrm{II})^2\} - \frac{2}{n-1}\mathrm{Tr}_{\bar{\eta}}(T_{\bar{\eta}}) + \frac{2(n+1)}{n-1}T_{\bar{\eta}}(F_{n+1}, F_{n+1}),$$

which gives the asserted formula immediately. The converse is easy to verify. □

Now we should verify that the constraints are indeed preserved under the evolution

$$\Delta(V, W) = \Delta(F_{n+1}, F_{n+1}) \cdot \bar{\eta}(V, W), \quad D_{\bar{\eta}}\psi = \lambda_Q \psi.$$

We note at this point that the divergence of the energy-momentum tensor $T_{\bar{\eta}}$ vanishes, $\mathrm{div}_{\bar{\eta}}(T_{\bar{\eta}}) = 0$, since $T_{\bar{\eta}}$ is defined by eigenspinors of the Dirac operator $D_{\bar{\eta}}$ (see [12]). Then, computing the divergence, $\mathrm{div}_{\bar{\eta}}(\Delta) = 0$, and expressing the covariant derivative $\nabla^{\bar{\eta}}$ in terms of ∇^{η}, we find that

$$\begin{aligned} 0 = &\sum_{j=1}^{n} (\mathrm{d}\Delta^H)(F_j)F_j + \sum_{j=1}^{n} (\nabla^{\eta}_{F_{n+1}} \Delta^M)(F_j)F_j - \Delta^H \sum_{i,j=1}^{n} \eta(\Lambda_g(E_i, E_i), E_j)F_j \\ &- (\mathrm{Tr}_{\bar{g}}\mathrm{II}) \sum_{j=1}^{n} \Delta^M(F_j)F_j - \Delta^H \sum_{i,j=1}^{n} \eta(E_i, \Lambda_g(E_i, E_j))F_j \\ &- \sum_{j=1}^{n} \Delta^M(\rho^{-1}\nabla^g_{F_j}\zeta + \rho^{-1}\Theta(F_j))F_j, \end{aligned}$$

and

$$\begin{aligned} 0 = &\sum_{j=1}^{n} (\nabla^{\eta}_{F_j} \Delta^M)(F_j)F_{n+1} + (\mathrm{d}\Delta^H)(F_{n+1})F_{n+1} + \sum_{j=1}^{n} \Delta^M((\nabla^g_{F_j} L^{-1})(E_j))F_{n+1} \\ &+ \sum_{j=1}^{n} \Theta(F_j, F_j)\{\rho\Delta^H + \Delta^M(\zeta)\}F_{n+1} \\ &- \sum_{j=1}^{n} \Delta^M(L^{-1}\{\Lambda_g(E_j, E_j)\})F_{n+1} + 2\rho^{-1} \sum_{j=1}^{n} \mathrm{d}\rho(F_j)\Delta^M(F_j)F_{n+1} \end{aligned}$$

where we have used the formula established in Proposition 2.1,

$$\nabla^{\bar{g}}_{F_i} F_j = L^{-1}(\nabla^g_{L^{-1}E_i} E_j) + L^{-1}\{\Lambda_g(E_i, E_j)\}.$$

Rewriting the above two equations with respect to η-orthonormal frame $(E_1, \ldots, E_n, E_{n+1})$, we arrive at a non-linear hyperbolic system of first-order differential equations of the form

$$\sum_{k=1}^{n} A(k) \cdot \nabla^{\eta}_{E_k} \Phi + B \cdot \nabla^{\eta}_{E_{n+1}} \Phi + C \cdot \Phi = 0,$$

where

$$\Phi = \left\{ \sum_{j=1}^{n} \Delta^M(E_j)E_j \right\} + (\Delta^H)E_{n+1}, \qquad A(k) = \begin{pmatrix} -\rho^{-1}\zeta^k (L^{-2})^i_j, & (L^{-2})^i_k \\ (L^{-2})^k_j, & -\rho^{-1}\zeta^k \end{pmatrix},$$

$$B = \begin{pmatrix} \rho^{-1}(L^{-2})^i_j, & 0 \\ 0, & \rho^{-1} \end{pmatrix},$$

and C is a $(1, 1)$-tensor field given by

$$C^i_j = -(\mathrm{Tr}_{\bar{g}}\mathrm{II})(L^{-2})^i_j - \rho^{-1}\sum_{u=1}^{n}(L^{-2})^{iu}\eta(\nabla^g_{E_u}\zeta + \Theta(E_u), E_j),$$

$$C^i_{n+1} = -\sum_{u,v=1}^{n}(L^{-1})^{iv}\{\eta(\Lambda_g(E_u, E_u), E_v) + \eta(E_u, \Lambda_g(E_u, E_v))\},$$

$$C^{n+1}_j = \sum_{u,v=1}^{n}(L^{-1})^{uv}\eta((\nabla^g_{E_u}L^{-1})(E_v), E_j) + \sum_{u,v=1}^{n}\zeta^j(L^{-2})^{uv}\Theta(E_u, E_v) - \sum_{u=1}^{n}\eta(L^{-1}\{\Lambda_g(E_u, E_u)\}, E_j) + \frac{2}{\rho}\sum_{u=1}^{n}(L^{-2})^u_j\,\mathrm{d}\rho(E_u),$$

$$C^{n+1}_{n+1} = \rho\sum_{u,v=1}^{n}(L^{-2})^{uv}\Theta(E_u, E_v).$$

We observed that the $(1, 1)$-tensor fields $A(k)$ and B are symmetric (with respect to reference metric η). Moreover, B is positive definite ($B \geq cI$ for some positive number $c > 0$), provided that every slice of Q^{n+1} is compact. Thus, it is shown that, under our evolution, the constraints are locally preserved. Note that, when we consider the warped product metrics as in Sections 5 and 7, the local preservation of constraints holds without the assumption that every slice of Q^{n+1} is compact.

Next, we state a complete set of evolution equations for the Einstein–Dirac equation. Soon we will also define the corresponding initial data sets precisely. Combining Corollary 3.1, Propositions 3.2, 4.1, 4.2 and 6.1 altogether, we easily obtain the evolution system of three differential equations, describing the evolution of metrics $L^2 = \bar{g}$, that of symmetric $(0, 2)$-tensor fields II and that of spinor fields ψ, respectively:

(E1) $$\eta((\nabla^\eta_{E_{n+1}}L)(V), L(W)) + \eta(L(V), (\nabla^\eta_{E_{n+1}}L)(W)) = \eta((\nabla^g_\zeta L)(V), L(W)) + \eta(L(V), (\nabla^g_\zeta L)(W)) + \eta(\nabla^g_V\zeta + \Theta(V) - \rho\mathrm{II}(V), L^2(W)) + \eta(L^2(V), \nabla^g_W\zeta + \Theta(W) - \rho\mathrm{II}(W)),$$

(E2) $$(\nabla^\eta_{E_{n+1}}\mathrm{II})(V, W) = \frac{\rho}{n-1}\{S_{\bar{g}} + \mathrm{Tr}_{\bar{g}}(\mathrm{II}^2) - (\mathrm{Tr}_{\bar{g}}\mathrm{II})^2\}\bar{g}(V, W) + \frac{\epsilon\rho}{4}(V\cdot\nabla^{\bar{g}}_W\psi + W\cdot\nabla^{\bar{g}}_V\psi, \psi) + \frac{\epsilon\rho}{8}(\{V\cdot\mathrm{II}(W) + W\cdot\mathrm{II}(V)\}\cdot F_{n+1}\cdot\psi, \psi) + \frac{\epsilon\lambda_Q\rho}{2(n-1)}(\psi, \psi)\bar{g}(V, W) - \frac{\epsilon\rho}{n-1}\left(\sum_{i=1}^{n}F_i\cdot\nabla^{\bar{g}}_{F_i}\psi, \psi\right)\bar{g}(V, W) + \rho\cdot\{-\mathrm{Ric}_{\bar{g}}(V, W) - 2\bar{g}(V, \mathrm{II}^2(W)) + (\mathrm{Tr}_{\bar{g}}\mathrm{II})\cdot\mathrm{II}(V, W)\} + (\nabla^{\bar{g}}_\zeta\mathrm{II})(V, W) + \bar{g}(\mathrm{II}(V), \nabla^{\bar{g}}_W\zeta) + \bar{g}(\mathrm{II}(W), \nabla^{\bar{g}}_V\zeta) + \bar{g}(\Theta(V), \mathrm{II}(W)) + \bar{g}(\Theta(W), \mathrm{II}(V)) + (\nabla^{\bar{g}}_W\,\mathrm{d}\rho)(V),$$

$$
\begin{aligned}
\text{(E3)}\quad \nabla^{\eta}_{E_{n+1}}\psi &= \nabla^{g}_{\zeta}\psi - \lambda_Q \rho E_{n+1}\cdot\psi + \frac{\rho}{2}(\mathrm{Tr}_{\bar g}\mathrm{II})\psi \\
&\quad + \rho E_{n+1}\cdot\left\{\sum_{i=1}^{n} E_i\cdot\nabla^{g}_{L^{-1}E_i}\psi + \frac{1}{4}\sum_{j,k,l=1}^{n}(\Lambda_g)_{jkl}E_j\cdot E_k\cdot E_l\cdot\psi\right\} \\
&\quad + \frac{1}{4}\sum_{i=1}^{n} E_i\cdot(\nabla^{\eta}_{E_{n+1}}L)(L^{-1}E_i)\cdot\psi + \frac{\rho}{4}\sum_{i=1}^{n}E_i\cdot(L\circ \mathrm{II}\circ L^{-1}) \\
&\quad \times (E_i)\cdot\psi - \frac{1}{4}\sum_{i=1}^{n}E_i\cdot(\nabla^{g}_{\zeta}L)(L^{-1}E_i)\cdot\psi \\
&\quad - \frac{1}{4}\sum_{i=1}^{n}E_i\cdot L(\nabla^{g}_{L^{-1}E_i}\zeta)\cdot\psi - \frac{1}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\Theta\circ L^{-1})(E_i)\cdot\psi \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\mathrm{d}\rho(L^{-1}E_i)E_i\cdot E_{n+1}\cdot\psi.
\end{aligned}
$$

In case of $n = 2m-1$, the equation (E3) may be certainly replaced by the one in Corollary 4.1, and the terms for $\mathrm{Tr}_{\bar\eta}(T_{\bar\eta})$ and $T_{\bar\eta}(F_{n+1}, F_{n+1})$ in (E2) may be replaced by the ones in Corollary 4.2.

Let us now define the initial data sets. We derive the constraint equations on initial hypersurfaces in a natural way, by combining Proposition 4.2 (Corollary 4.2) with the relations

$$T_{\bar\eta}(V, F_{n+1}) = \mathrm{Ric}_{\bar\eta}(V, F_{n+1}) = \mathrm{d}(\mathrm{Tr}_{\bar g}\mathrm{II})(V) - \mathrm{div}_{\bar g}(\mathrm{II})(V),$$

and

$$T_{\bar\eta}(F_{n+1}, F_{n+1}) = \mathrm{Ric}_{\bar\eta}(F_{n+1}, F_{n+1}) - \tfrac{1}{2}S_{\bar\eta} = \tfrac{1}{2}\{-S_{\bar g} + (\mathrm{Tr}_{\bar g}\mathrm{II})^2 - \mathrm{Tr}_{\bar g}(\mathrm{II}\circ\mathrm{II})\}.$$

Definition 6.1 (In case of $n = 2m$). An *initial data set* $(M^{2m}, \bar g, \mathrm{II}_M, \psi_M)$ for the Einstein–Dirac equation on Q^{2m+1} consists of a slice M^{2m} with, defined on it, a metric $\bar g$, a symmetric $(0, 2)$-tensor field II_M and a spinor field ψ_M satisfying the *momentum constraint*

$$
\begin{aligned}
&\mathrm{d}(\mathrm{Tr}_{\bar g}\mathrm{II}_M)(V) - \mathrm{div}_{\bar g}(\mathrm{II}_M)(V) \\
&\quad = \tfrac{1}{4}\epsilon((\sqrt{-1})^{m+1}\mu_{\bar g}\cdot\{\nabla^{\bar g}_V\psi_M - V\cdot D_{\bar g}\psi_M\}, \psi_M), \quad \epsilon = \pm 1,
\end{aligned}
$$

as well as the *Hamiltonian constraint*

$$-S_{\bar g} + (\mathrm{Tr}_{\bar g}\mathrm{II}_M)^2 - \mathrm{Tr}_{\bar g}(\mathrm{II}_M\circ\mathrm{II}_M) = -\epsilon(D_{\bar g}\psi_M - \lambda_Q\psi_M, \psi_M), \quad \lambda_Q \in \mathbb{R}.$$

Definition 6.2 (In case of $n = 2m-1$). An *initial data set* $(M^{2m-1}, \bar g, \mathrm{II}_M, \psi^+_M, \varphi^+_M)$ for the Einstein–Dirac equation on Q^{2m} consists of a slice M^{2m-1} with, defined on it, a metric $\bar g$, a symmetric $(0, 2)$-tensor field II_M and two-spinor fields ψ^+_M, φ^+_M satisfying the *momentum constraint*

$$
\begin{aligned}
&\mathrm{d}(\mathrm{Tr}_{\bar g}\mathrm{II}_M)(V) - \mathrm{div}_{\bar g}(\mathrm{II}_M)(V) \\
&\quad = \tfrac{1}{4}\epsilon(\nabla^{\bar g}_V\psi^+_M - V\cdot D_{\bar g}\psi^+_M, \varphi^+_M) - \tfrac{1}{4}\epsilon(\nabla^{\bar g}_V\varphi^+_M - V\cdot D_{\bar g}\varphi^+_M, \psi^+_M),
\end{aligned}
$$

as well as the *Hamiltonian constraint*

$$-S_{\bar g}+(\mathrm{Tr}_{\bar g}\mathrm{II}_M)^2-\mathrm{Tr}_{\bar g}(\mathrm{II}_M\circ\mathrm{II}_M)$$
$$=-\epsilon(D_{\bar g}\psi_M^+,\varphi_M^+)-\epsilon(D_{\bar g}\varphi_M^+,\psi_M^+)+\epsilon\lambda_Q\{(\psi_M^+,\psi_M^+)+(\varphi_M^+,\varphi_M^+)\},\quad \lambda_Q\in\mathbb{R}.$$

7. A local existence theorem

For a specific class of initial data sets, we will establish a local existence theorem for the Einstein–Dirac equation. Let us begin with the case $n=2m$. Let $\psi_M=\psi_M^+ + \psi_M^-$ be a spinor field on (M^{2m},g_M) with $\psi_M^\pm\in\Gamma(\Sigma^\pm(M))$. Let $\Gamma_{\mathrm{odd}}(\psi_M)$ denote the space of all spinor fields of the form $\psi=h^+\psi_M^+ + h^-\psi_M^-$ on $Q^{2m+1}=M^{2m}\times\mathbb{R}$ defined by

$$\psi(x,t)=h^+(t)\psi_M^+(x)+h^-(t)\psi_M^-(x),\quad (x,t)\in M^{2m}\times\mathbb{R},$$

where $h^\pm:\mathbb{R}\to\mathbb{C}$ are complex-valued functions. The following lemma is an immediate consequence of Proposition 4.2, combined with Lemma 5.1.

Lemma 7.1. *Let $\psi_M=\psi_M^+ + \psi_M^-$ be a real Killing spinor on (M^{2m},g_M) with*

$$\nabla_V^{g_M}\psi_M^\pm=-\frac{\lambda_M}{2m}V\cdot\psi_M^\mp,\quad \lambda_M\in\mathbb{R}.$$

Then $(\psi_M,\psi_M)=(\psi_M^+,\psi_M^+)+(\psi_M^-,\psi_M^-)$ is constant on M^{2m}, and the energy-momentum tensor, determined by $\bar\eta\in\mathrm{WP}(g_M;a)$ and $\psi\in\Gamma_{\mathrm{odd}}(\psi_M)$, is given by

$$\mathrm{Tr}_{\bar\eta}(T_{\bar\eta})=\frac{\epsilon\lambda_Q}{2}\{(h^+\psi_M^+,h^+\psi_M^+)+(h^-\psi_M^-,h^-\psi_M^-)\},$$
$$T_{\bar\eta}(V,W)=\frac{\epsilon\lambda_M}{4m}\,\mathrm{e}^{f/2}\{(h^+\psi_M^-,h^-\psi_M^-)+(h^-\psi_M^+,h^+\psi_M^+)\}\eta(V,W),$$
$$T_{\bar\eta}(V,F_{2m+1})=-\frac{(2m+1)\epsilon\lambda_M}{8m}(h^+E_{2m+1}\cdot V\cdot\psi_M^-,h^-\psi_M^-)-\frac{(2m+1)\epsilon\lambda_M}{8m}(h^-E_{2m+1}\cdot V\cdot\psi_M^+,h^+\psi_M^+),$$
$$T_{\bar\eta}(F_{2m+1},F_{2m+1})=-\frac{\epsilon\lambda_M}{2}\,\mathrm{e}^{-f/2}\{(h^+\psi_M^-,h^-\psi_M^-)+(h^-\psi_M^+,h^+\psi_M^+)\}+\frac{\epsilon\lambda_Q}{2}\{(h^+\psi_M^+,h^+\psi_M^+)+(h^-\psi_M^-,h^-\psi_M^-)\}.$$

Proposition 7.1. *For $\bar\eta\in\mathrm{WP}(g_M;a)$ and $\psi\in\Gamma_{\mathrm{odd}}(\psi_M)$, the evolution equations* (E1)–(E3) *for the Einstein–Dirac equation are equivalent to*

(i) $$f_{tt}=\frac{af_tf_t}{2}-\frac{2}{m^2}(\lambda_M)^2\,\mathrm{e}^{(a-1)f}-\frac{\epsilon\lambda_Q}{2m-1}\,\mathrm{e}^{af}\langle h^+,h^+\rangle(\psi_M,\psi_M)+\frac{2m+1}{4m(2m-1)}\epsilon\lambda_M\,\mathrm{e}^{(a-1/2)f}\{\langle h^+,h^-\rangle+\langle h^-,h^+\rangle\}(\psi_M,\psi_M),$$

$$\text{(ii)}\quad h_t^+ = -\frac{m}{2} f_t h^+ + (\sqrt{-1})^{2m+3}\lambda_Q\, \mathrm{e}^{(a/2)f} h^+ - (\sqrt{-1})^{2m+3}\lambda_M\, \mathrm{e}^{(1/2)(a-1)f} h^-,$$

$$h_t^- = (\sqrt{-1})^{2m+3}\lambda_M\, \mathrm{e}^{(1/2)(a-1)f} h^+ - \frac{m}{2} f_t h^- - (\sqrt{-1})^{2m+3}\lambda_Q\, \mathrm{e}^{(a/2)f} h^-,$$

where we have used the notation $\langle , \rangle$ (*for complex-valued functions*) *to mean the standard Hermitian product.*

Proof. Proposition 4.1 implies that $\psi = h^+\psi_M^+ + h^-\psi_M^- \in \Gamma_{\mathrm{odd}}(\psi_M)$ satisfies the Dirac equation,

$$D_{\bar\eta}\psi = \lambda_Q \psi, \qquad \bar\eta \in \mathrm{WP}(g_M; a),$$

on $(Q^{2m+1}, \bar\eta)$ if and only if the second part (ii) of the proposition is true. It remains to verify that the first part (i) of the proposition is locally equivalent to the evolution equations (E1)–(E2). Substituting Lemmas 5.1, 5.2 and 7.1 in Proposition 6.1, we obtain

$$f_{tt} = \frac{af_t f_t}{2} - \frac{2}{m^2}(\lambda_M)^2\, \mathrm{e}^{(a-1)f} - \frac{\epsilon\lambda_Q}{2m-1}\, \mathrm{e}^{af}\{(h^+\psi_M^+, h^+\psi_M^+) + (h^-\psi_M^-, h^-\psi_M^-)\}$$
$$+ \frac{2m+1}{2m(2m-1)}\epsilon\lambda_M\, \mathrm{e}^{(a-1/2)f}\{(h^+\psi_M^-, h^-\psi_M^-) + (h^-\psi_M^+, h^+\psi_M^+)\}.$$

Now we must note that the second part (ii) of the proposition implies

$$\frac{d}{\mathrm{d}t}\{\langle h^+, h^+\rangle - \langle h^-, h^-\rangle\} = -mf_t\{\langle h^+, h^+\rangle - \langle h^-, h^-\rangle\}.$$

Therefore, provided $h^+(0) = h^-(0)$ holds initially, the equality $\langle h^+, h^+\rangle = \langle h^-, h^-\rangle$ is valid locally in t, and hence

$$(h^+\psi_M^+, h^+\psi_M^+) + (h^-\psi_M^-, h^-\psi_M^-) = \langle h^+, h^+\rangle(\psi_M, \psi_M),$$

is valid locally in t. Moreover, it is evident that

$$(h^+\psi_M^-, h^-\psi_M^-) + (h^-\psi_M^+, h^+\psi_M^+) = \tfrac{1}{2}\{\langle h^+, h^-\rangle + \langle h^-, h^+\rangle\}(\psi_M, \psi_M).$$

Thus we complete the proof of the proposition. □

Let $\mathrm{Re}(h^\pm)$ and $\mathrm{Im}(h^\pm)$ denote the real and imaginary part of the complex-valued functions $h^\pm$, respectively. Then, we observed that, if we take

$$\Psi = (f, f_t, \mathrm{Re}(h^+), \mathrm{Im}(h^+), \mathrm{Re}(h^-), \mathrm{Im}(h^-)),$$

as a set of six unknowns, then the system of evolution equations in Proposition 7.1 reduces to an autonomous equation

$$\frac{d}{\mathrm{d}t}\Psi = H(\Psi)$$

for some vector field H defined on the six-dimensional Euclidean space $\mathbb{R}^6$. This fact implies that, to each initial data, there corresponds a unique smooth local solution to the evolution system in Proposition 7.1.

Proposition 7.2. *Let (M^{2m}, g_M) be a Riemannian manifold admitting a real Killing spinor ψ_M. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists an open interval $(-\omega, \omega) \subset \mathbb{R}$ and a warped product metric $\bar{\eta}$ on $Q^{2m+1} = M^{2m} \times (-\omega, \omega)$ such that $(Q^{2m+1}, \bar{\eta})$ admits an Einstein spinor ψ to eigenvalue λ_Q. In particular, if ψ_M is a parallel spinor, then the Einstein spinor ψ coincides with the WK-spinor in Proposition 5.2.*

Proof. Let $\psi_M = \psi_M^+ + \psi_M^-$ be a real Killing spinor, to Killing number $-(\lambda_M/2m) \in \mathbb{R}$, on the initial hypersurface (M^{2m}, g_M). We identify M^{2m} with the subspace $M^{2m} \times \{0\} \subset M^{2m} \times \mathbb{R}$. Let $\bar{\eta} = \mathrm{e}^f \left(\sum_{i=1}^{2m} E^i \otimes E^i\right) + \mathrm{e}^{af}\,\mathrm{d}t \otimes \mathrm{d}t \in \mathrm{WP}(g_M; a)$ and $\psi = h^+\psi_M^+ + h^-\psi_M^- \in \Gamma_{\mathrm{odd}}(\psi_M)$ satisfy the initial conditions

$$h^+(0) = h^-(0) = 1, \qquad f(0) = 0,$$

and

$$f_t(0) = \pm\sqrt{\frac{4(\lambda_M)^2}{m^2} + \frac{2\epsilon(\lambda_Q - \lambda_M)}{m(2m-1)}(\psi_M, \psi_M)},$$

where we can always control $\epsilon = \pm 1$ and $(\psi_M, \psi_M) = \text{constant}$ so that

$$\frac{4(\lambda_M)^2}{m^2} + \frac{2\epsilon(\lambda_Q - \lambda_M)}{m(2m-1)}(\psi_M, \psi_M),$$

is non-negative. Let $\mathrm{II}_M = -\frac{1}{2} f_t(0) g_M$ be the symmetric $(0, 2)$-tensor field required to prescribe initial data. Then, with the help of Lemma 7.1, one verifies that the initial data $(M^{2m}, g_M, \mathrm{II}_M, \psi_M)$ satisfies the constraint equations in Definition 6.1. Moreover, we know that the evolution system in Proposition 7.1 is an autonomous equation and hence allows a local solution satisfying the initial data. This proves the former part of the proposition. Let us now suppose that the spinor ψ_M is a parallel spinor ($\lambda_M = 0$). In this case, we may assume that $\psi_M = \psi_M^+ \in \Gamma(\Sigma^+(M))$ and $\psi = h^+\psi_M^+$, and hence the evolution system in Proposition 7.1 simplifies to

$$f_{tt} = \frac{af_t f_t}{2} - \frac{\epsilon\lambda_Q}{2m-1}\,\mathrm{e}^{af}\langle h^+, h^+\rangle(\psi_M^+, \psi_M^+),$$
$$h_t^+ = -\frac{m}{2} f_t h^+ + (\sqrt{-1})^{2m+3}\lambda_Q\,\mathrm{e}^{(a/2)f} h^+.$$

On the other hand, since ψ is (locally) an Einstein spinor, the equation

$$\mathrm{Ric}_{\bar{\eta}}(F_{2m+1}, F_{2m+1}) - \tfrac{1}{2} S_{\bar{\eta}} = T_{\bar{\eta}}(F_{2m+1}, F_{2m+1}),$$

gives

$$\frac{m(2m-1)}{4}\,\mathrm{e}^{-af} f_t f_t = \frac{\epsilon\lambda_Q}{2}\langle h^+, h^+\rangle(\psi_M^+, \psi_M^+).$$

Thus, it follows that the function f must satisfy $f_{tt} + \frac{1}{2}(m-a) f_t f_t = 0$ whose solutions are exactly the ones given in Lemma 5.3, with $n = 2m$. □

Now, we proceed to the other case $n = 2m - 1$. Let φ_M^+ be a spinor field on M^{2m-1} (note that we have identified $\Sigma(M)$ with $\Sigma^+(Q)$). Let $\Gamma_{\mathrm{even}}(\varphi_M^+)$ denote the space of all spinor fields of the form $\varphi = h^+\varphi_M^+ + k^+E_{2m}\cdot\varphi_M^+$ on $Q^{2m} = M^{2m-1}\times\mathbb{R}$ defined by

$$\varphi(x,t) = h^+(t)\varphi_M^+(x) + k^+(t)E_{2m}\cdot\varphi_M^+(x), \quad (x,t)\in M^{2m-1}\times\mathbb{R},$$

where $h^+, k^+ : \mathbb{R}\to\mathbb{C}$ are complex-valued functions. The following lemma is an immediate consequence of Corollary 4.2 combined with Lemma 5.1.

Lemma 7.2. *Let φ_M^+ be a real Killing spinor on (M^{2m-1}, g_M) with*

$$\nabla_V^{g_M}\varphi_M^+ = -\frac{\lambda_M}{2m-1}V\cdot E_{2m}\cdot\varphi_M^+.$$

Then $(\varphi_M^+,\varphi_M^+)$ is constant on M^{2m-1}, and the energy-momentum tensor, determined by $\bar\eta\in \mathrm{WP}(g_M;a)$ and $\varphi\in\Gamma_{\mathrm{even}}(\varphi_M^+)$, is given by

$$\begin{aligned}
\mathrm{Tr}_{\bar\eta}(T_{\bar\eta}) &= \tfrac{1}{2}\epsilon\lambda_Q\{(h^+\varphi_M^+, h^+\varphi_M^+) + (k^+\varphi_M^+, k^+\varphi_M^+)\},\\
T_{\bar\eta}(V,W) &= \frac{\epsilon\lambda_M}{2m-1}\,\mathrm{e}^{f/2}(h^+\varphi_M^+, k^+\varphi_M^+)\eta(V,W),\\
T_{\bar\eta}(V,F_{2m}) &= -\frac{m\epsilon\lambda_M}{2m-1}(h^+V\cdot E_{2m}\cdot\varphi_M^+, k^+\varphi_M^+),\\
T_{\bar\eta}(F_{2m},F_{2m}) &= -\epsilon\lambda_M\,\mathrm{e}^{-(f/2)}(h^+\varphi_M^+, k^+\varphi_M^+)\\
&\quad + \tfrac{1}{2}\epsilon\lambda_Q\{(h^+\varphi_M^+, h^+\varphi_M^+) + (k^+\varphi_M^+, k^+\varphi_M^+)\}.
\end{aligned}$$

Proposition 7.3. *For $\bar\eta\in\mathrm{WP}(g_M;a)$ and $\varphi\in\Gamma_{\mathrm{even}}(\varphi_M^+)$, the evolution equations* (E1)–(E3) *for the Einstein–Dirac equation are equivalent to*

$$\begin{aligned}
\text{(i)}\quad f_{tt} &= \frac{af_tf_t}{2} - \frac{8}{(2m-1)^2}(\lambda_M)^2\,\mathrm{e}^{(a-1)f}\\
&\quad - \frac{\epsilon\lambda_Q}{2(m-1)}\,\mathrm{e}^{af}\{(h^+\varphi_M^+, h^+\varphi_M^+) + (k^+\varphi_M^+, k^+\varphi_M^+)\}\\
&\quad + \frac{2m}{(m-1)(2m-1)}\epsilon\lambda_M\,\mathrm{e}^{(a-(1/2))f}(h^+\varphi_M^+, k^+\varphi_M^+),
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad h_t^+ &= -\frac{2m-1}{4}f_th^+ - \lambda_M\,\mathrm{e}^{(1/2)(a-1)f}h^+ + \lambda_Q\,\mathrm{e}^{(a/2)f}k^+,\\
k_t^+ &= -\lambda_Q\,\mathrm{e}^{(a/2)f}h^+ - \frac{2m-1}{4}f_tk^+ + \lambda_M e^{(1/2)(a-1)f}k^+,
\end{aligned}$$

where we may choose h^+, k^+ to be real-valued functions.

Proof. Corollary 4.1 implies that $\varphi = h^+\varphi_M^+ + k^+E_{2m}\cdot\varphi_M^+\in\Gamma_{\mathrm{even}}(\varphi_M^+)$ satisfies the Dirac equation

$$D_{\bar\eta}\varphi = \lambda_Q\varphi, \quad \bar\eta\in\mathrm{WP}(g_M;a),$$

on $(Q^{2m}, \bar{\eta})$ if and only if the second part (ii) of the proposition is true, where h^+, k^+ may be chosen to be real-valued functions. Substituting Lemmas 5.1, 5.2 and 7.2 in Proposition 6.1, we obtain the first part (i) of the proposition. □

With the help of Propositions 7.2 and 7.3, we prove the main theorem of the paper.

Theorem 7.1. *Let (M^n, g_M) be a Riemannian manifold admitting a real Killing spinor φ_M. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists an open interval $(-\omega, \omega) \subset \mathbb{R}$ and a warped product metric $\bar{\eta}$ on $Q^{n+1} = M^n \times (-\omega, \omega)$ such that $(Q^{n+1}, \bar{\eta})$ admits an Einstein spinor φ to eigenvalue λ_Q. In particular, if φ_M is a parallel spinor, then the Einstein spinor φ coincides with the WK-spinor in Theorem 5.1.*

Proof. Because of Proposition 7.2, it suffices to prove the theorem for the case $n = 2m-1$. Let φ_M^+ be a real Killing spinor, to Killing number $-(\lambda_M/2m-1) \in \mathbb{R}$, on the initial hypersurface (M^{2m-1}, g_M). Let $\bar{\eta} = \mathrm{e}^f \left(\sum_{i=1}^{2m-1} E^i \otimes E^I\right) + \mathrm{e}^{af}\,\mathrm{d}t \otimes \mathrm{d}t \in \mathrm{WP}(g_M; a)$ and $\varphi = h^+\varphi_M^+ + k^+E_{2m} \cdot \varphi_M^+ \in \Gamma_{\mathrm{even}}(\varphi_M^+)$ satisfy the initial conditions

$$h^+(0) = k^+(0) = 1, \qquad f(0) = 0,$$

and

$$f_t(0) = \pm\sqrt{\frac{16(\lambda_M)^2}{(2m-1)^2} + \frac{4\epsilon(\lambda_Q - \lambda_M)}{(m-1)(2m-1)}(\varphi_M^+, \varphi_M^+)},$$

where we can always control $\epsilon = \pm 1$ and $(\varphi_M, \varphi_M) =$ constant so that

$$\frac{16(\lambda_M)^2}{(2m-1)^2} + \frac{4\epsilon(\lambda_Q - \lambda_M)}{(m-1)(2m-1)}(\varphi_M^+, \varphi_M^+),$$

is non-negative. Let $\mathrm{II}_M = -\frac{1}{2} f_t(0) g_M$ be the symmetric $(0, 2)$-tensor field required to prescribe initial data. Then, with the help of Lemma 7.2, one verifies that the initial data set $(M^{2m}, g_M, \mathrm{II}_M, \psi_M^+ = \varphi_M^+, \varphi_M^+)$ satisfies the constraint equations in Definition 6.2. Moreover, as in the case of $n = 2m$, we find that there exists a unique local solution to the evolution system in Proposition 7.3 satisfying the initial data. One proves the latter part of the theorem in a similar way as in the proof for Proposition 7.2. □

The Einstein spinors of Theorem 7.1 do not generally extend to $M^n \times \mathbb{R}$, since the evolution system in Proposition 7.1 (resp. Proposition 7.3), in general, do not allow global solutions. However, via reparametrization $(-\omega, \omega) \to \mathbb{R}$, we conclude that, indeed, there exist global solutions to the Einstein–Dirac equation on $M^n \times \mathbb{R}$.

Corollary 7.1. *Let (M^n, g_M) be a Riemannian manifold admitting a real Killing spinor. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists a warped product metric $\bar{\eta}^*$ on $Q^{n+1} = M^n \times \mathbb{R}$ such that $(Q^{n+1}, \bar{\eta}^*)$ admits an Einstein spinor to eigenvalue λ_Q.*

Proof. We consider the case $n = 2m$. The same argument is valid for the other case $n = 2m-1$. By Theorem 7.1, there exists a solution $(\bar{\eta}, \psi)$ to the Einstein–Dirac equation on $M^{2m} \times (-\omega, \omega)$ for some positive number ω, with $\bar{\eta} = \mathrm{e}^{f} \left(\sum_{i=1}^{2m} E^i \otimes E^i\right) + \mathrm{e}^{af} \mathrm{d}t \otimes \mathrm{d}t \in \mathrm{WP}(g_M; a)$ and $\psi = h^+ \psi_M^+ + h^- \psi_M^- \in \Gamma_{\mathrm{odd}}(\psi_M)$. Let $\gamma : \mathbb{R} \to (-\omega, \omega)$ be a diffeomorphism, e.g., defined by

$$\gamma(s) = \frac{2\omega}{\pi} \arctan(s), \quad s \in \mathbb{R}.$$

Now we pullback the metric $\bar{\eta}$ as well as the Einstein spinor ψ to $M^{2m} \times \mathbb{R}$ via the diffeomorphism

$$I \times \gamma : M^{2m} \times \mathbb{R} \to M^{2m} \times (-\omega, \omega), \qquad (x, s) \mapsto (x, \gamma(s)).$$

In fact, using the diffeomorphism γ and the relations

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{2\omega}{\pi(s^2+1)}, \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\pi}{2\omega}(s^2+1),$$

we can explicitly express the pullbacked objects as

$$\bar{\eta}^* := (I \times \gamma)^*(\bar{\eta}) = \mathrm{e}^{f^*} \left(\sum_{i=1}^{2m} E^i \otimes E^i\right) + \mathrm{e}^{af^*} \frac{4\omega^2}{\pi^2(s^2+1)^2} \mathrm{d}s \otimes \mathrm{d}s,$$

where $f^*(s) = (f \circ \gamma)(s)$, and

$$\psi^* := (I \times \gamma)^*(\psi) = (h^+ \circ \gamma)\psi_M^+ + (h^- \circ \gamma)\psi_M^-.$$

Obviously, $(\bar{\eta}^*, \psi^*)$ is a global solution to the Einstein–Dirac equation on $M^{2m} \times \mathbb{R}$. □

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